Permutation Puzzles

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Interpreting the 15 Puzzle and the Rubik's Cube using Group Theory

The 15 puzzle consists of a 4×4 grid with 15 tiles, numbered 1 to 15 and an empty slot, 16. The aim is to order the numbered tiles, starting from a jumbled position, by sliding the tiles into the empty slot. Using group theory, one can interpret configurations and moves of the puzzle as permutations e.g by multiplying a configurations permutation with a moves permutation a new permutation will be produced.The set $\{1, .., 16\}$ can be used to keep track of the puzzle pieces and positions, hence the puzzle group is a subgroup of S_{16}.

One can also interpret configurations and moves of the Rubik's Cube as permutations, see figure (c). Other concepts considered include inverses, identities and commutators and the permutations of a Rubik's Cube form a group, denoted R.

15 Puzzle: Impossible Position

Applying a move M to a configuration, C, changes the puzzle's configuration to the product MC in S_{16}. Hence, if we apply the moves $M_1, M_2, .., M_r$ to C then the final configuration is $M_r, .., M_2M_1C$.

The Impossible Position has configuration $(14, 15)$, see figure (a) and the solved 15 puzzle is the identity, (1).

Hence, we have $C = (1), C' = (14, 15) \in S_{16}$ so there exists some transpositions $\Gamma_1, \Gamma_2, .., \Gamma_r \in S_{16}$ $s.t$

$$(14, 15) = \Gamma_1, \Gamma_2, .., \Gamma_r(1).$$

Since, in $C' = (14, 15)$, the empty space remains in the same location it must have moved an even amount of times and since it changes position with every Γ_i the right hand side of the equation is even. However, C' is not an even permutation, hence this is a contradiction.

The 15 Puzzle is generated by the 3-cycles $(11, 12, i)$

The 15 puzzle group is A_{15}.

For $n \geq 3$, A_n is generated by the 3-cycles $(1, 2, i)$ but in A_{15} the common pair of terms are 11 and 12.

Hence, the 15 puzzle is generated by $(11, 12, i)$.

These 3-cycles can then be used to generate all other 3-cycles of A_{15}.

Let $\sigma = (11, 12, i)$ and $\tau = (11, 12, j)$ then

$$\sigma\tau\sigma^{-1} = (12, i, j)$$

$$\Rightarrow (12, i, k)(12, j, i)(12, k, i) = (i, j, k) = (i, j)(j, k)$$

Hence, the set of permutations obtained by moving the blank space produces all possible permutations of A_{15}. So all even permutations of the puzzle can be obtained, but odd ones cannot. This enforces why the move $(14, 15)$ is impossible.

Impossible Position

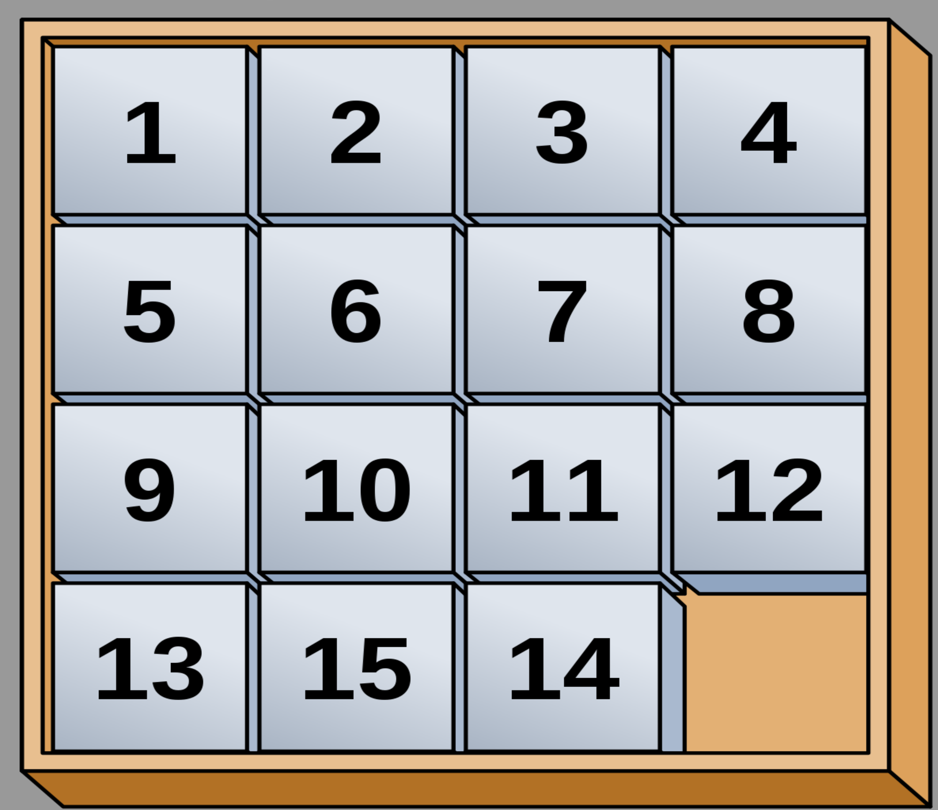

Figure (a) from https://math.stackexchange.com

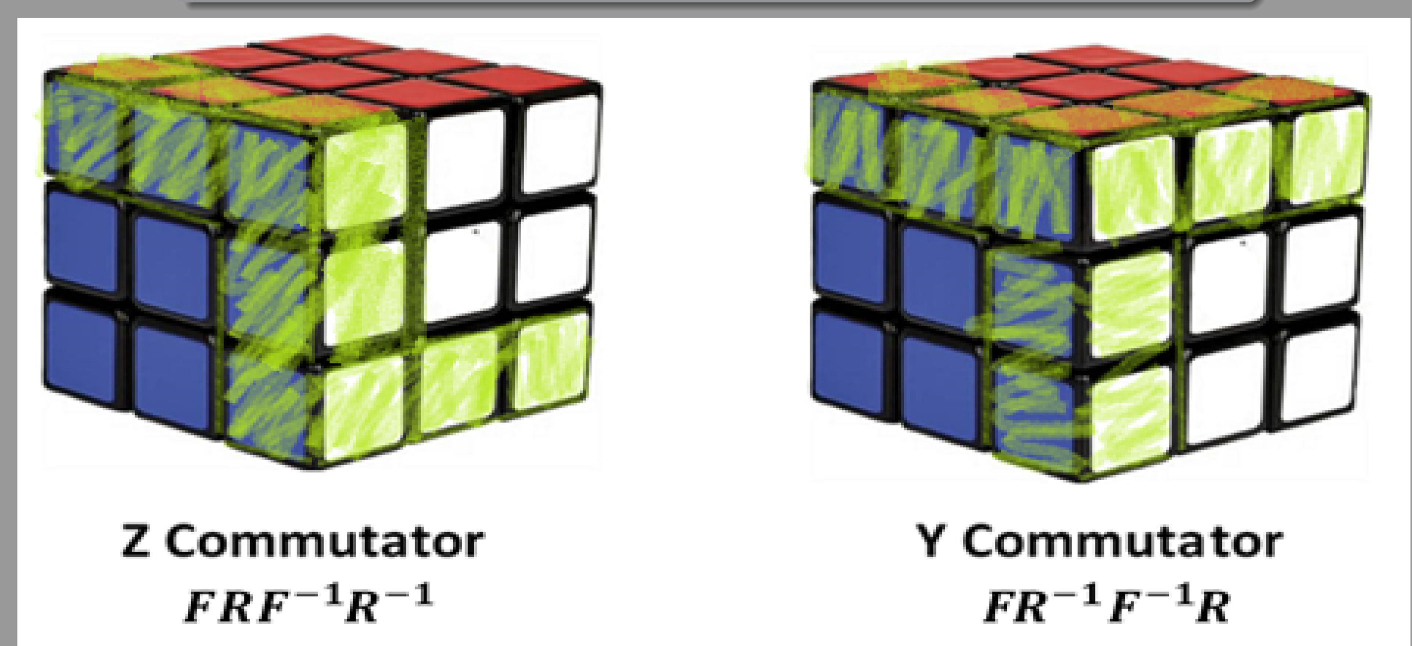

Figure (b) from https://www.pricerunner.com (annotations by the author)

Cubic Group

Closure: If moves $M_1, M_2 \in R, M_1 * M_2 \in R$.

Associativity: If $M_1 * M_2$ is performed, followed by M_3 it is the same as performing a move M_1 followed by $M_2 * M_3$.

Identity: No physical movements of cube.

Inverse: For every move, $M \in R$, $\exists M^{-1} \in R$, s.t. $M * M^{-1}$ results in the initial position.

Permutation Moves

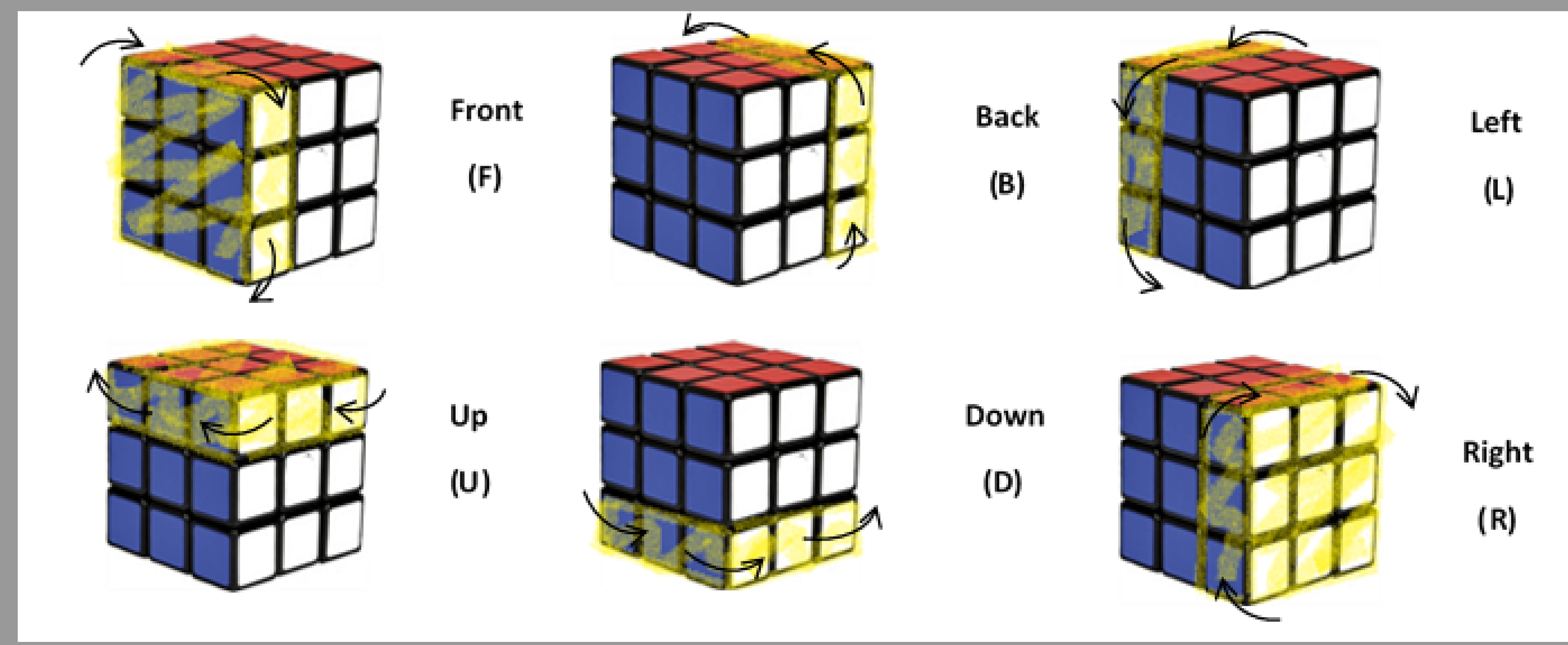

Figure (c) from https://www.pricerunner.com (annotations by the author)

$$F = (flu, fur, frd, fdl)(fu, fr, fd, fl)$$
$$B = (bru, bul, bld, bdr)(bu, bl, bd, br)$$
$$L = (lbu, luf, lfd, ldb)(lu, lf, ld, lb)$$
$$R = (rfu, rub, rbd, rdf)(ru, rb, rd, rf)$$
$$U = (ulb, ubr, urf, ufl)(ub, ur, uf, ul)$$
$$D = (dlf, dfr, drb, dbl)(df, dr, db, dl)$$

Untwisting of Corner Cubies

The use of commutators is the best way to only disturb the position of specific pieces, the most useful being the Z and Y commutators. For some element X and Y

Z commutator - $[X, Y] = XYX^{-1}Y^{-1}$.

Y commutator - $[X, Y^{-1}] = XY^{-1}X^{-1}Y$ (see figure (b)).

Suppose you want to twist corner cubies, one clockwise and one anticlockwise, you will use the Z commutator.

Let $X = L^{-1}D^2LBD^2B^{-1}$ and $Y = U$ - X.Then

$$[X, Y] = L^{-1}D^2LBD^2B^{-1}UL^{-1}D^2LBD^2B^{-1}U^{-1}$$

produces the permutation $(ulb)_-(ufl)_+$ which is a clockwise twist of ufl and an anticlockwise twist of (ulb).

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