Finite Element Boundary Value Problems

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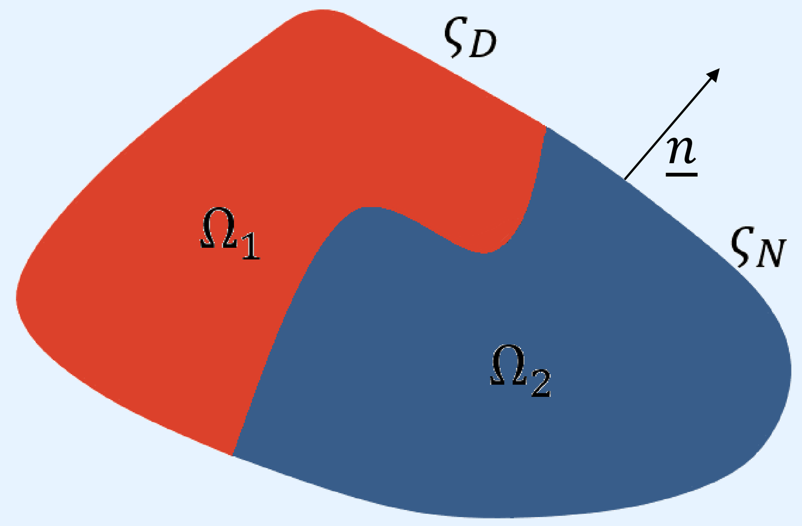
Introduction:

The general form of a boundary value problem is the Strong Form. Where Ω is the domain, f is the source term, a is the material constant and u is the unknown we want to find. The boundary $\partial\Omega$ is formed by $\varsigma_N \cup \varsigma_D$.

$$\nabla \cdot a\nabla u = -f \text{ in } \Omega$$
$$u = u_D \text{ on } \varsigma_D$$
$$\underline{n} \cdot a\nabla u = g_N \text{ on } \varsigma_N$$
$$\partial\Omega = \varsigma_D \cup \varsigma_N$$



ς_D represents the Dirichlet part of the boundary, which the value of the function is defined on the boundary and ς_N represents the Neumann boundary where the normal derivative of the function on the boundary is specified.

Applications:

Boundary value problems are applicable to real world physical applications such as electrostatics, magnetostatics (2D), heat conduction and potential flow.

Application	a	u	f
Electrostatics	ε	ϕ	ρ_V
Magnetostatics (2D)	μ^{-1}	A_Z	J_Z
Heat Conduction	K	T	$\frac{\partial T}{\partial t}$
Potential Flow	1	ψ	0

Strong Form to Weak Form:

To establish the Weak Form, multiply the governing equation by a weight function w, which is chosen to vanish on $\partial\Omega$, and integrate over Ω. Then use integration by parts to reduce the order of the highest derivatives and obtain the form:

Find u $\in \{H^1(\Omega),\ u = u_D \text{ on } \varsigma_D\}$ where $H^1(\Omega) = \{u \in L_2(\Omega),\ \nabla u \in (L^2(\Omega))^d\}$ such that

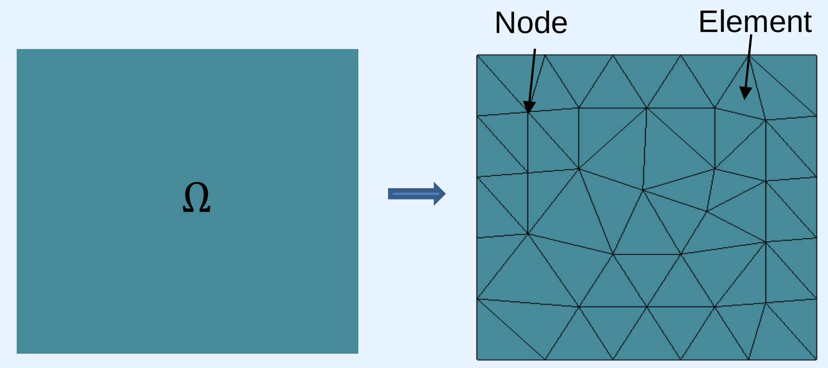
$$\int_\Omega a\nabla u \cdot \nabla w \, d\Omega = \int_{\varsigma_N} \underline{n} \cdot a\nabla u \, ds + \int_\Omega fw \, d\Omega, \quad \forall w \in H_0^1(\Omega)$$

Advantages of Finite Elements Method:

- Computers are highly efficient and accurate at using the Finite element method, so can be used to approximate solutions quickly.
- The method can be used on complex geometries
- Finite elements is adaptable by using H and P refinement.
- Low investment compared to creating a prototype to run tests on and after a simulation the model can be easily modified based on results.

Method of Finite Elements:

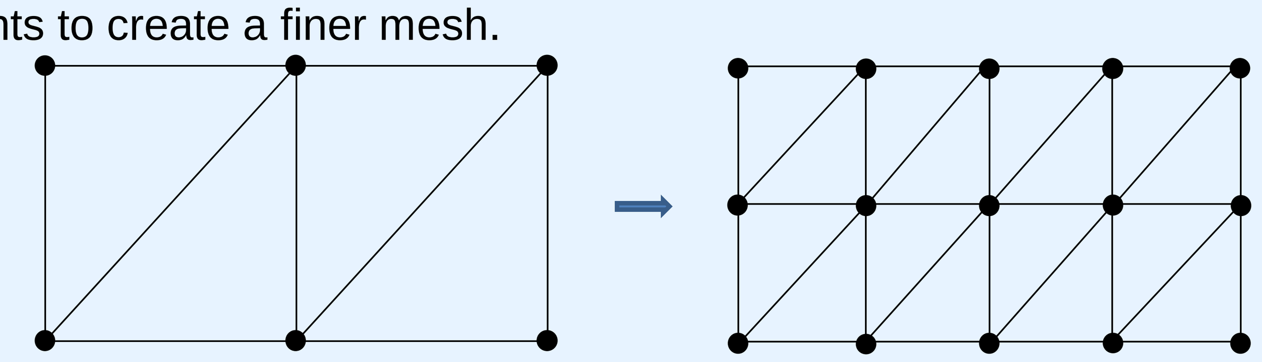
Finite elements (FE) is based on the idea of discretizing the bounded domain into a finite number of smaller non-overlapping subdomains called elements connected by nodes.



Then shape functions are chosen to interpolate between nodes. A linear system of equations is obtained and can then be solved. The accuracy of the approximation can be improved using H and P refinement.

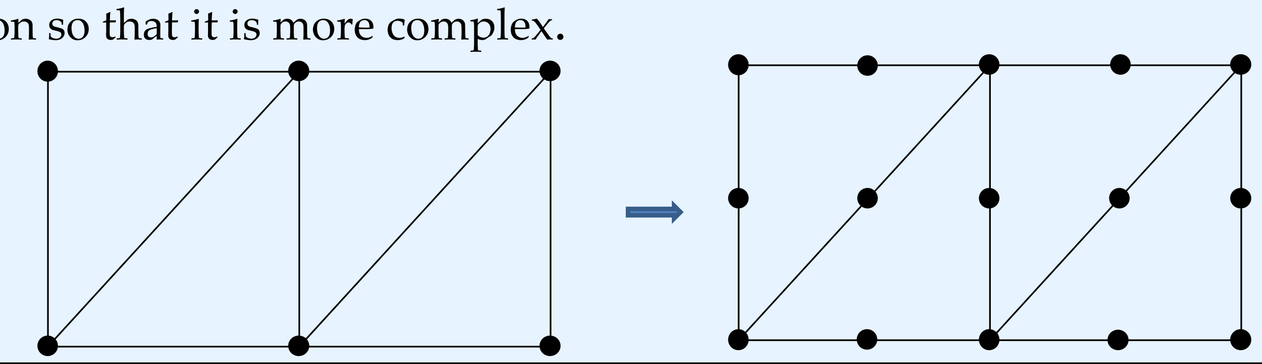
H-Refinement:

The H-method improves the accuracy of results by increasing the number of elements to create a finer mesh.



P-Refinement:

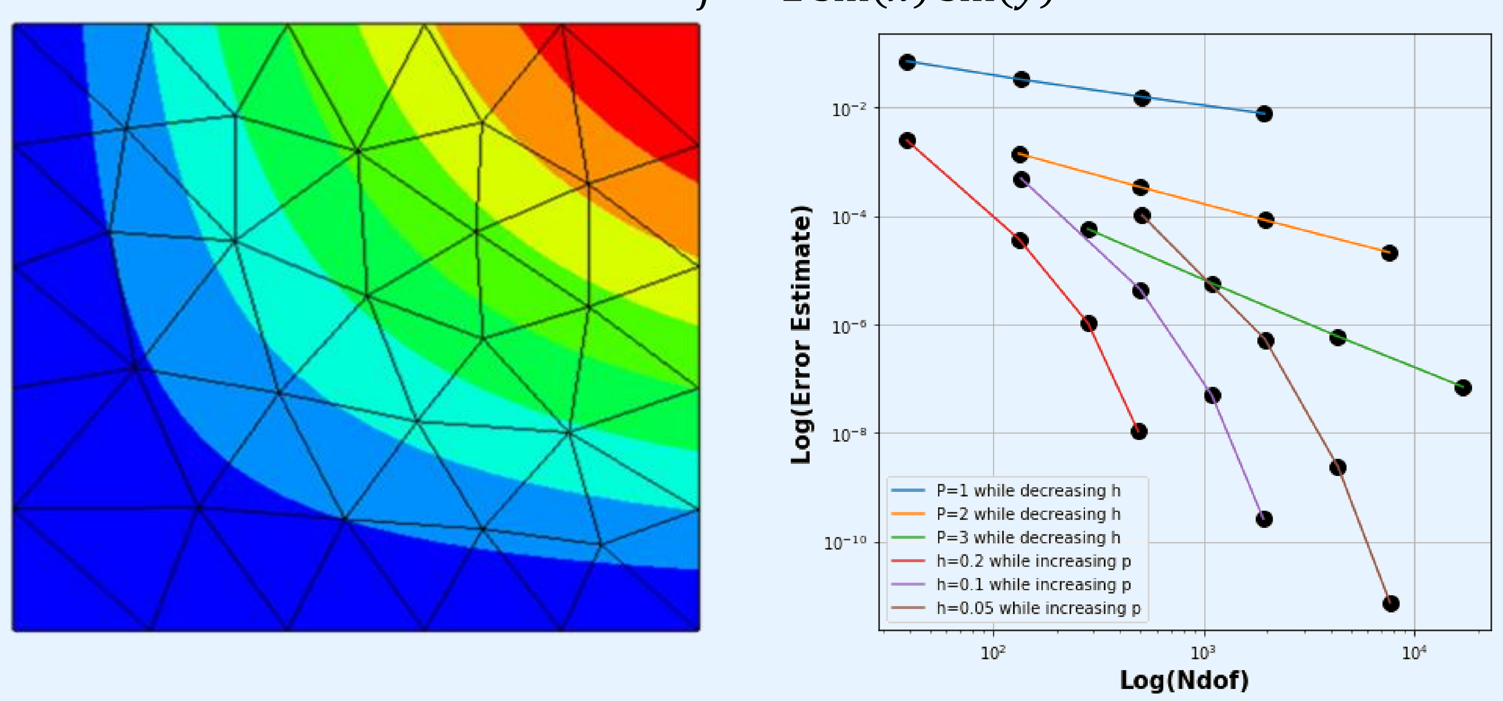
The P-method improves the accuracy results by increasing order of the shape function so that it is more complex.



Example of a Smooth Boundary value Problem:

$\Omega = [0,1]^2$ with pure Dirichlet boundary conditions according to the exact solution u,

$$u(x,y) = \sin(x)\sin(y)$$
$$a = 1$$
$$f = 2\sin(x)\sin(y)$$

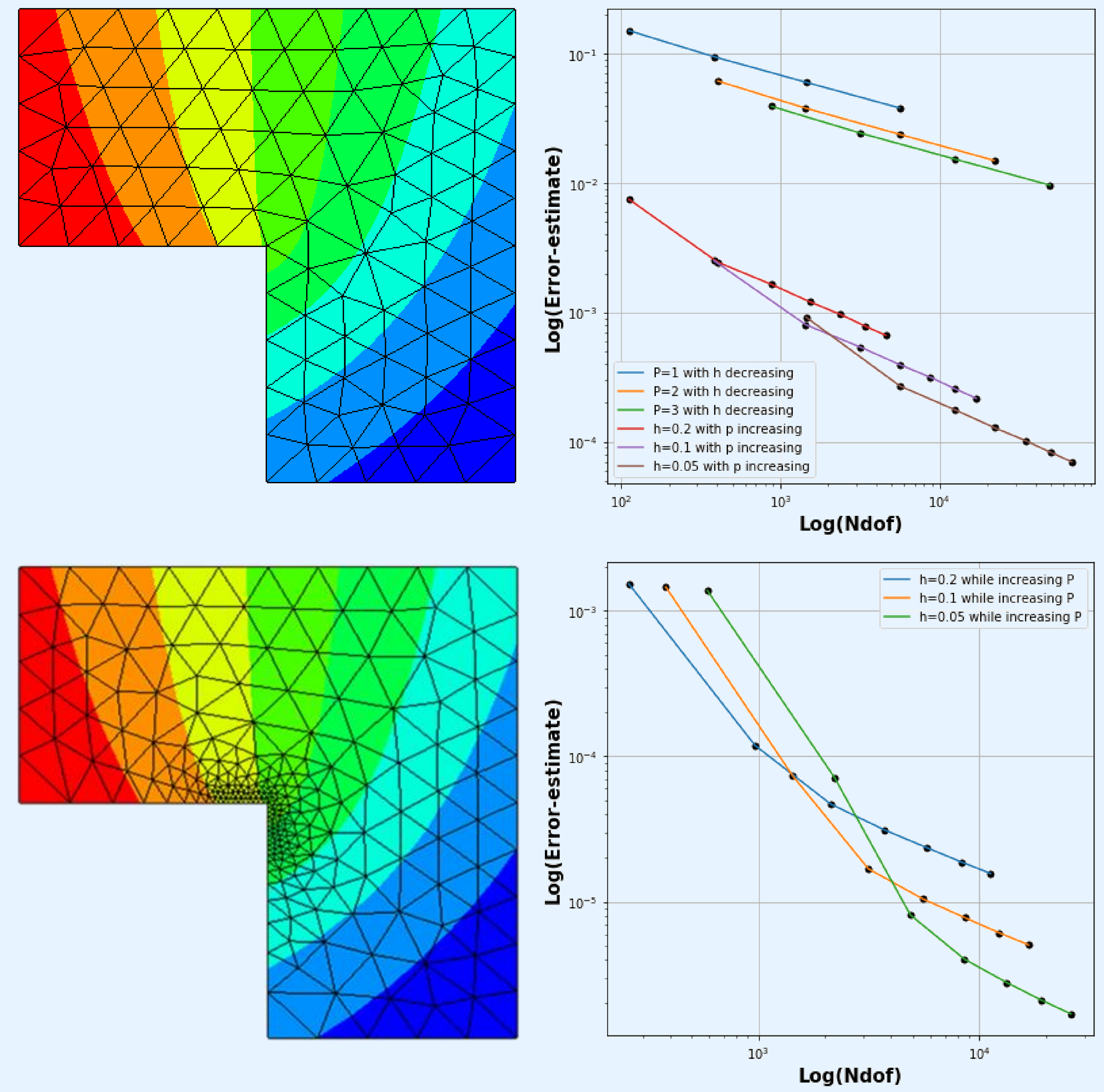


We can see from the results of $\|e\|_{\{L_2(\Omega)\}} = \left(\int_\Omega |u - u_{hp}|^2 d\Omega\right)^{1/2}$ that P-refinement leads to an exponential convergence rate, while H-refinement leads to algebraic convergence rate.

Example of a Non-smooth Problem:

We set $\Omega = [-1,1]^2$ with pure pure Dirichlet boundary conditions according to the exact solution in polar coordinates,

$$u(r,\theta) = r^{2/3}\sin\left(\frac{4\pi}{3} - \frac{2\theta}{3}\right)$$
$$a = 1$$
$$f = 0$$



In the case of a Non-Smooth Boundary problem a combination of both H and P refinement is needed to achieve an exponential convergence rate.

Discussions and Conclusion:

To conclude FEM can approximate solutions to Boundary Value problems with a high degree of accuracy. For smooth problems, the most efficient use of computational power is to use P-refinement. While for non-smooth problems a combination of both H and P refinement is needed in order to achieve the exponential convergence rate to the approximation.

The next step is to apply FEM to boundary value problems that do not have an analytical solution. As FEM is a numerical method, we should be able to approximate the solution.

References:

NGSolve [Online]. 2019. [Accessed 5th October 2020]. Available from: https://ngsolve.org/

Düster, A., Rank, E. and Szabó, B. (2017). The p-Version of the Finite Element and Finite Cell Methods. In Encyclopedia of Computational Mechanics Second Edition (eds E. Stein, R. Borst and T.J.R. Hughes). Available from: https://doi.org/10.1002/9781119176817.ecm2003g