LONG NONLINEAR SURFACE AND INTERNAL GRAVITY WAVES IN A ROTATING OCEAN

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Abstract. Nonlinear dynamics of surface and internal waves in a stratified ocean under the influence of the Earth's rotation is discussed. Attention is focussed upon guided waves long compared to the ocean depth. The effect of rotation on linear processes is reviewed in detail as well as the existing nonlinear models describing weakly and strongly nonlinear dynamics of long waves. The influence of rotation on small-scale waves and two-dimensional effects are also briefly considered. Some estimates of the influence of the Earth's rotation on the parameters of real oceanic waves are presented and related to observational and numerical data.

1. Introduction

Waves in the oceans are such a complicated and diverse phenomenon that even their mere classification is sometimes a difficult problem. Even if the fluid compressibility is neglected (i.e. no acoustic waves) and electromagnetic effects are excluded from consideration, various types of waves remain which are commonly distinguished by the type of their "restoring force" (gravity, capillarity, Coriolis, meridional variation of the Coriolis force, etc). This well-known classification distinguishes surface gravity and capillary waves, internal waves, inertial-gyroscopic waves, Rossby waves, etc. [42]. Further, barotropic motions such as surface waves and depth-independent Rossby waves are distinguished from baroclinic ones such as internal waves and depth-dependent Rossby waves, the former being largely uninfluenced by the density stratification, while the latter are essentially supported by the density stratification. Moreover, even within the same class the wave dynamics may differ essentially depending on their scale.

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An “observational” approach first distinguishes motions by their spatial and temporal scales. Naturally, the most thoroughly studied are the two most energetic parts of the spectrum corresponding to the two extreme cases.

The first is the large-scale geostrophic (or quasi-geostrophic) motions for which the latitudinal gradient of the Coriolis force, the so-called “beta-effect”, is the main factor. They cover the scales of order and higher than the Rossby – Obukhov radius, which is defined as the scale for a balance between the inertial and rotational terms. Its value is about 10^3 km for barotropic motions in the ocean and the atmosphere, and at least one order smaller for baroclinic motions [35]. Their characteristic periods exceed a day and often are much larger than that. Synoptic oceanic eddies are a well-known example of this type of motion. Being the primary target of basin-scale circulation studies in the ocean, these motions are by now relatively well understood. This understanding has been accumulated in a large number of monographs, surveys and textbooks on geophysical fluid dynamics (see, e.g., [56, 19, 35].

The second case is the relatively small-scale motions with the wavelengths from millimeters to a few hundreds of meters and and periods from a fraction of a second to an hour, such as, for example, wind waves and shortscale internal waves in the ocean. For them, gravity and capillary forces are dominant while the Earth’s rotation usually plays a negligible role. There is also a tremendous bulk of literature concerning such motions, although some crucial questions still remain open [60, 72, 42, 46].

Here we shall discuss another class of oceanic wave motions which has been less thoroughly investigated, especially with respect to their nonlinear aspects. For them, gravity and rotation are the main governing forces, and their effects are often of comparable importance while the beta-effect does not play a considerable role. They typically have length scales less than the corresponding Rossby radius, and a time scale less than the inertial period. Thus, the beta-effect is typically unimportant for them, whereas the Coriolis term itself plays a definitive role together with inertial terms, and the buoyancy terms for internal gravity waves. For the latter in the ocean these scales lie in the range of a few kilometers to a few dozens kilometers. Also long surface waves, such as tsunamis, having lengths up to a few hundred kilometers, may be relevant here. Of course, there is already accummulated a large bulk of measurements of wave spectra in the ocean (such as the Garrett – Munk spectrum for the oceanic internal waves, see, e.g., [14, 60, 42] and the quite recent spatial spectra of inertial waves derived from satellite altimeter data, e.g., [22, 23]) which include these ranges, but that kind of spectra gives just a “climate-type” description, i.e. strongly averaged over extremely large time and space scales. For a proper understanding of the mechanisms of wave evolution (especially in the upper layer and marginal zones of the ocean) this kind of data usually proves to be insufficient and even may deliver a misleading message concerning the importance of the motions under consideration. For instance, internal waves of the scales under consideration are not necessarily the most pronounced in climatic spectra, but often are the prevailing motions in the continental shelf zone, where they play an important role in exchange processes between the deep and shallow waters.

Tsunamis' space-time scales, although absent in climatic spectra of surface waves, still have an obvious motivation for their study. Only relatively recently have the dynamical properties of such waves begun to attract more attention [58]. Note also that recent field studies being able to cover larger spatial and temporal intervals have shown that rotational effects may be dynamically important even for motions of smaller scales, in particular for the formation of groups of internal solitary waves by barotropic tides at the shelf margin (e.g.,[15]). In this connection, even earlier theoretical studies of the effect of rotation on gravity waves may be of more interest now for geophysical applications.

In this review we shall try to summarize our present understanding in this domain, focusing our attention on the interplay of nonlinearity and dispersion. We shall also discuss basic physical effects and estimate their role in typical oceanic conditions. In most cases these waves in the ocean are still long compared to the characteristic vertical scales, such as the depth or, for internal waves, to a characteristic scale of the density stratification. Hence, a modal approach is the most appropriate: the waves are represented as a sum of vertical normal modes in a layered medium. Therefore they are often referred to as "guided" or "ducted" waves to emphasize the prevailing modal character of their propagation in contrast to the atmospheric internal waves of comparable scales where the time of formation of the waveguide is often longer than that of the dynamic processes under consideration. In such situations it is appropriate to consider wave propagation for arbitrary angles to the horizontal, and even in the vertical direction as well.

Nevertheless, many of the theoretical results reviewed below can also be applied to atmospheric motions as well, although we do not consider here the specific details for the atmosphere. Note, however, that in the atmosphere (and sometimes in the ocean) smaller-scale motions propagating at some arbitrary angle to the vertical direction, may "feel" the rotation. On the other hand, the internal wave modes considered here can be (at least for the weakly nonlinear cases) represented as a superposition of waves propagating in oblique directions to the vertical and being reflected or refracted at the boundaries of a waveguide (that is the essence of Brillouin's representation of propagating waves in a waveguide). From this viewpoint our approach is quite general.

The paper is organized as follows. After an outline of the linear properties of the waves considered (which, however, contains some new elements regarding the asymptotic properties of dispersive waves), we shall concentrate on nonlinear effects. We shall outline the main model equations which adequately describe the wave class considered, and give the constraints following from these equations. Then we will discuss the main types of analytical and numerical solutions currently available for weakly nonlinear waves with two types of interplaying dispersive effects: "low-frequency" (or Coriolis) dispersion due to rotation and "high-frequency" (or Boussinesq) dispersion due to non-hydrostatic effects (i.e., influence of the finite depth). Some results are also given for the strongly nonlinear case which are exact for long surface waves in the case when the high-frequency dispersion can be neglected. Also two-dimensional waves with smoothly curved fronts and the effect of rotation on short waves are briefly discussed; in the latter case, the rotation can be important for the nonlinear description of long wave packets. Finally, we briefly consider the problem of internal wave excitation by tides and discuss some observational data.

BASIC EQUATIONS AND ASSUMPTIONS

We shall base our theoretical considerations on the Euler equations for an inviscid, incompressible, density stratified fluid in a rotating frame. Although the effects of viscosity might be essential in some cases, for most of the problems we are interested in its role is usually less important than that of the effects of nonlinearity and dispersion.

As the intrinsic wave dynamics proves to be rich enough in itself, we shall drop out of our consideration the interaction of the scales of interest with background motions of larger scales, and shall focus on the wave motions *per se*. Thus we designate the perturbations due to the waves to be $\mathbf{u}(u, v, w)$, p, ρ relative to a background field at rest $p_0(z)$, $\rho_0(z)$ which is assumed to be independent of time and horizontal coordinates. The Euler equations for these perturbations in a Cartesian frame, with the z-axis directed upward and $xy-$plane lying on the unperturbed surface are,

$$(\rho_0 + \rho)D_t\mathbf{u} + (\rho_0 + \rho)[2\Omega \times \mathbf{u}] + \nabla p - \rho\mathbf{g} = \mathbf{0},$$

$$D_t\rho + w d_z\rho_0 = 0, \tag{1}$$

$$(\nabla \cdot \mathbf{u}) = 0.$$

Here $\mathbf{u} = (\mathbf{q}, w) = (u, v, w)$, p, and ρ are the velocity, pressure and density perturbations, respectively, $\rho_0(z)$ is the equilibrium density distribution, $\nabla = (\nabla_\perp, \partial_z) = (\partial_x, \partial_y, \partial_z)$ and $D_t = \partial_t + (\mathbf{u} \cdot \nabla)$ are the gradient operator and the material derivative respectively, and Ω is the vector of angular velocity of the Earth's rotation.

Two commonly adopted simplifying approximations are helpful for further analysis. The first is the Boussinesq approximation, i.e. the density variations ρ, being small, are retained only in the buoyancy term of the momentum equation, while the vertical variations of ρ_0 are taken into account only through a single characteristic, the bouyancy frequency N (see, e.g., [60, 42, 46]), where

$$N^2 = -\frac{g}{\rho_0} d_z\rho_0$$

In fact, this approximation will not be made during the description of basic wave equations, and is usually only invoked to obtain simple analytical expressions.

The second is the so-called “traditional approximation” allowing one to neglect the horizontal components of Ω. It is justified by the smallness of vertical velocities in comparison to horizontal ones.* In view of that we introduce the local Coriolis parameter f in the traditional manner: $f = 2(\Omega \cdot \nabla z) = 2\Omega \sin\phi$, where ϕ is local latitude. At moderate latitudes (around at $\phi = 45°$), $f \simeq 10^{-4}\ s^{-1}$ and we shall use this value in further numerical estimates.

Let us supplement the system (1) with inviscid boundary conditions (given for simplicity only for oceanic surface and internal waves) which are specified as follows:

$$p_0 + p|_{z=\eta} = 0, \qquad D_t\eta = w|_{z=\eta}, \qquad w|_{z=-H} = 0, \tag{2}$$

where η is the sea level elevation, and H is the undisturbed ocean depth. In the case of internal waves under the Boussinesq approximation the free surface boundary conditions reduce with sufficient accuracy to the simple homogeneous condition $w = 0$ at the unperturbed surface $z = 0$.

2. Preliminary Discussion of Linear Waves

2.1. Dispersion properties

It is natural before proceeding to any consideration of the properties of nonlinear waves that we are mostly interested in here, to begin with some preliminary analysis of the linear motions. From this we expect to get some understanding of the basic dynamical processes.

The linearization of the system (1) can be presented in the form of the single equation,

$$(D_t^2 + f^2)\partial_z(\rho_0\partial_z w) + (D_t^2 + N^2)\nabla_\perp^2(\rho_0 w) = 0, \tag{3}$$

where $D_t = \partial_t$ is the linearized version of the convective operator, and $\nabla_\perp^2$ is the horizontal Laplace operator. The boundary conditions (2) can also be linearized, and expressed in terms of the single variable w,

$$\begin{aligned} g\nabla_\perp^2 w - (D_t^2 + f^2)\partial_z w = 0 \qquad &\text{at} \quad z = 0, \\ w = 0 \qquad &\text{at} \quad z = -H. \end{aligned} \tag{4}$$

As we have indicated in the introduction, our primary focus here is on long waves propagating horizontally. Hence it is appropriate to develop a modal approach,

* A rigorous estimate of the range of validity of the latter approximation is a quite nontrivial task (for detailed analysis see [36, 28]), although it is always straightforward to check *a posteriori* the effect of neglected terms.

for which the horizontal spatial and temporal uniformity of these linearized equations allows us to perform a Fourier transform. Thus we are led to the Sturm – Liouville boundary-value problem for each harmonic component of the form $F(z)\exp[i(kx + ly - \omega t)]$, which determines the dispersion relation, i.e. the dependence of ω on $\mathbf{k} = (k, l)$, and the vertical structure of the modal functions. The corresponding boundary-value problem in terms of the vertical velocity [for the structural function $W(z)$], is described by,

$$\begin{aligned} \partial_z(\rho_0 \partial_z W) + \frac{N^2 - \omega^2}{\omega^2 - f^2}(k^2 + l^2)\rho_0 W &= 0, \\ \partial_z W - \frac{k^2 + l^2}{\omega^2 - f^2} g W &= 0 \qquad \text{at} \quad z = 0, \\ W &= 0 \qquad \text{at} \quad z = -H. \end{aligned} \tag{5}$$

It is convenient to consider first the simplest model of stratification, that of constant Brunt – Väisälä frequency N. The cases of surface and internal waves then might be treated in a similar manner. The difference is that the surface mode (a “zero-number mode”) does not have any vertical oscillations in the interior, the vertical velocity decays monotonically from a finite value at the surface to zero at the bottom, while the internal modes have n zeros, specified by the mode number $n \geq 1$. Note that a more general form of dispersion relation for this model which includes also the horizontal component of the Coriolis force can be found in [42]. Its derivation is straightforward but the analysis of the cumbersome implicit relations is tedious. Hence we confine ourselves here to the case when the traditional approximation holds. For both surface and internal waves the dispersion relation can be presented in the implicit form of a transcendental equation. The horizontal isotropy of the modal system (5) allows us to consider waves propagating just in the x-direction. Thus the disprsion relation takes the form

$$\omega^2 - f^2 = \frac{gk}{\nu} \tanh(\nu k H), \tag{6}$$

where $\nu^2 = -(N^2 - \omega^2)/(\omega^2 - f^2)$. We assume the most important case when $N > f$, although sometimes the inverse inequality takes place in the ocean, especially in the near-equatorial zones.

Surface waves usually have frequencies far exceeding N although situations where $\omega < N$ can occur as well. Internal waves always have frequencies lower than N. Being mostly interested in long waves with $kH \ll 1$ we can easily determine the asymptotic behaviour of the dispersion curves. In the absence of both rotation and stratification the dispersion of long surface waves is governed by the classical shallow-water expansion for a wave propagating in the positive direction,

$$\omega \simeq c_0 k - \beta k^3, \tag{7}$$

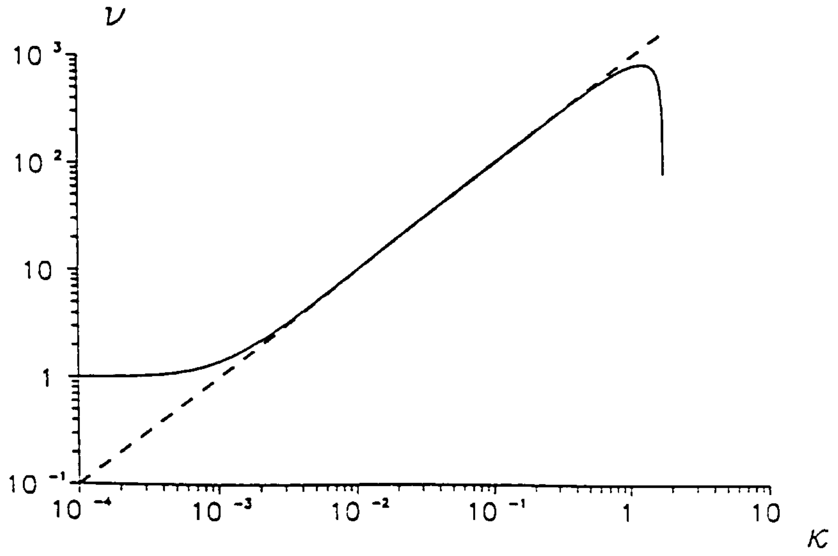

Figure 1. The dispersion relation (8) for waves in a rotating ocean, plotted in dimensionless variables: $\nu^2 = 1 + (1 - \kappa^2/3)\kappa^2/\mathrm{Ro}$, where $\nu = \omega/f$; $\kappa = kH$; $\mathrm{Ro} = f^2 H/g$. Here $\mathrm{Ro} = 10^{-6}$.

where $c_0 = \sqrt{gH}$ is the velocity of the longest waves without rotation and $\beta = c_0 H^2/6$ is the dispersion parameter. This type of dispersion is often called *negative*, as the waves with higher wavenumbers have lower phase velocities. Taking rotation into account but still supposing that $\omega^2 \gg f^2$, so that $\nu \approx 1$, we have

$$\omega^2 \simeq c_0^2 k^2 + f^2 - 2c_0\beta k^4. \tag{8}$$

This dispersion relation has an inflection point (Figure 1). Note that the phase velocity $c_{ph} \equiv \omega/k$ decreases monotonically with k, while the group velocity $c_g \equiv d_k\omega$ has a maximum (and the value $d_k^2\omega$ changes its sign) at a finite value of $k = k_{in} \equiv \sqrt{f/(c_0 H)}$. One can immediately get from (8) a generalization of (7) under the further assumption $f \ll c_0 k$:

$$\omega \simeq c_0 k + \frac{f^2}{2c_0 k} - \beta k^3 \tag{9}$$

If we take $f = 10^{-4}\,s^{-1}$, $H = 4 \cdot 10^3\,m$, so that $c_0 \approx 200\,m/s$, we get for k_{in} a value of about $10^{-5}\,m^{-1}$ which corresponds to a wavelength of about $560\,km$. In shallow water, say $H = 40\,m$, we get $k_{in} \approx 3.5 \cdot 10^{-4}\,m^{-1}$ with the corresponding wavelength of $18\,km$. These estimates can be important from the viewpoint of both linear and nonlinear theory. We note that (8) has a wider range of validity than (9), which requires $\omega \gg f$ and thus $c_0^2 k^2 \gg f^2$, that in terms of spatial scales yields the condition $k \gg k_* \equiv f/c_0$ (note that c_0/f is the Rossby – Obukhov radius). We recall that requirement of weak nonhydrostatic dispersion bounds the range of allowed scales from above, $k \ll k_\dagger \equiv 1/H$. For the open ocean ($H \simeq 4 \cdot 10^4\, m$) the gap is rather narrow: $2\pi/k_* \simeq 60\, km$ ($k_* \simeq 10^{-4}\, m^{-1}$), $2\pi/k_\dagger \simeq 25\, km$ ($k_\dagger \simeq 2.5 \cdot 10^{-4}\, m^{-1}$). However for shallow seas and coastal zones it expands considerably, since $k_\dagger$ increases more rapidly then k_*. For low latitudes the long-wave bound k_* is relaxed.

The influence of rotation can be important here when it becomes comparable with nonhydrostatic dispersion due to finite depth. The corresponding spatial scale, the wavelength λ_r, equals $2\pi/k_{in} = 2\pi\sqrt{c_0 H/f}$, that is a few hundreds of kilometers for the open ocean and only a few tens of kilometers in shallow seas. Thus these characteristic scales are realistic for, say, tsunami waves. Note that thus defined these rotation scales are still much smaller than the corresponding Rossby – Obukhov radius c_0/f which is equal to $2 \cdot 10^3\, km$ at moderate latitudes. The validity of (9) in preference to the more general expression (8) requires that $k_{in} \gg k_*$, or that $\lambda_r \ll 2\pi c_0/f$. Here the ratio k_*/k_{in} is $\sqrt{Hf/c_0}$ which is much smaller than unity in oceanic conditions.

When stratification is taken into account for surface waves of very low frequencies, comparable to N or even smaller, it slightly increases c_0, while preserving the negative character of dispersion.

The frequencies of *internal waves* are always less than N and their dispersion relation is given by the other roots of (6) which may be easily found from the simplified version of it corresponding to the "rigid lid" approximation mentioned above, so that

$$\tan\left(kH\sqrt{\frac{N^2-\omega^2}{\omega^2-f^2}}\right) = 0, \quad \text{or} \quad kH\sqrt{\frac{N^2-\omega^2}{\omega^2-f^2}} = \pm n\pi \quad (n = 1, 2, \ldots). \tag{10}$$

The explicit formula takes the form

$$\omega^2 = \frac{N^2(kH)^2 + f^2(\pi n)^2}{(\pi n)^2 + (kH)^2} \tag{11}$$

and eventually for long waves ($kH \ll \pi n$) we come to the universal form for a long wave dispersion relation, identical to (9), where now $c_0 = NH/(\pi n)$ is the phase velocity in the limit of zero rotation and infinitely long waves, and the dispersion coefficient $\beta = c_0 H^2/2(\pi n)^2$. An estimate of the rotation scale now yields $\lambda_r \approx 2.63(H/n)\sqrt{N/f}$, while the Rossby – Obukhov radius is again c_0/f, or here $NH/\pi n f$. Here the ratio $k_*/k_{in} \simeq 1.3\sqrt{f/N}$ and is independent of the depth H and the mode number n. For typical oceanic values of N (e.g. $10^{-3}\, s^{-1}$) this ratio is 0.4 in mid-latitudes. Consequently here (8) is perhaps more appropriate than (9) which requires that $\lambda_r \ll 2\pi c_0/f$, or $k_*/k_{in} \ll 1$. For $H = 4 \cdot 10^3\, m$, λ_r is about $33\, km$ for the first mode, and decreases with the mode number as n^{-1}.

To provide a more complete picture, a brief consideration of some alternative basic simple models is useful. For instance the widely used model of a two-layer fluid bounded from above and below by a rigid lid with two vertically uniform layers of different densities $\rho_1, \ \rho_2$ and thicknesses H_1, H_2, yields the dispersion relation,

$$\omega\sqrt{\omega^2 - f^2} = (\rho_2 - \rho_1)gk\left[\rho_2 \coth\frac{\omega H_2 k}{\sqrt{\omega^2 - f^2}} + \rho_1 \coth\frac{\omega H_1 k}{\sqrt{\omega^2 - f^2}}\right]^{-1}, \quad (12)$$

which in the limit $kH_1, \ kH_2 \ll 1$ has asymptotics similar to (8) with the only difference lying in the specific dependence of the coefficients c_0, β on the stratification parameters,

$$c_0 = \left[\frac{g(\rho_2 - \rho_1)H_1H_2}{\rho_2 H_1 + \rho_1 H_2}\right]^{1/2} \simeq \left[\frac{g(\rho_2 - \rho_1)}{\rho_2}\frac{H_1H_2}{H_1 + H_2}\right]^{1/2},$$

$$\beta = \frac{c_0 H_1 H_2}{6}\left(\frac{\rho_1 H_1 + \rho_2 H_2}{\rho_2 H_1 + \rho_1 H_2}\right) \simeq \frac{c_0 H_1 H_2}{6}.$$

The latter approximate equalities are valid under the assumption $\rho_1 \simeq \rho_2$. We stress that the dispersion is again always negative, i.e. β is positive, within this model. If the thicknesses of the layers are strongly separated, say $H_1 \ll H_2$, and $kH_1 \ll 1$, while $kH_2 \gg 1$, an important intermediate asymptotic relation occurs

$$\omega^2 = f^2 + c_0^2 k^2 - \rho_2 c_0^2 H_1 k^2 \mid k \mid / \rho_1, \quad (13)$$

where now $c_0 = \sqrt{(\rho_2 - \rho_1)gH_1/\rho_1}$.

Typical horizontal spatial scales for these asymptotics to occur depend strongly on the scales of stratification and fall in the range of around a hundred meters for the common situations in the ocean. And again, the dispersion is negative. Here $k_{in} = \sqrt{f/c_0 H^*}$ where $H^* = \sqrt{H_1 H_2}$, and so the ratio $k_*/k_{in} = \sqrt{H^* f/c_0} = \sqrt{f(H/g')^{\frac{1}{2}}}$ where g' is the reduced gravity, $g' = g(\rho_2 - \rho_1)/\rho_2$. Note that this ratio is independent of the depth ratio H_1/H_2, and is much less than unity for typical oceanic conditions.

Although these formulae are based on two simple models of stratification they still give an adequate picture of many basic properties of real processes and can provide reasonable estimates for the relevant physical scales.

2.2. Evolution of wave packets

Next we shall consider nonstationary processes associated with the evolution of quasisinusoidal wave trains due to dispersion. Certainly, for this, linear case the answer can be found just by use of the Fourier transform of the initial perturbation. However, a more visual way is to study the wave evolution in space and time by representing it as a modulated wavetrain with slowly varying amplitude, frequency, and wavenumber [51, 74]. At the first glance, such a solution may seem to be rather special but actually, as we shall see, any localized perturbation of small amplitude will eventually transform into a modulated wavetrain. Thus, we consider solutions of the form $u = A(x,t)e^{i\theta}$, where A is a slowly varying function of its arguments, and the instantaneous values of frequency and wavenumber,

$$\omega(x,t) = -\partial_t\theta, \qquad k(x,t) = \partial_x\theta \tag{14}$$

are also slowly varying functions satisfying the ”local” dispersion equation $\omega = \omega(k)$, or, in our case (8). Then, by definition,

$$\partial_x\omega + \partial_t k = 0, \tag{15}$$

or

$$\partial_t k + c_g\partial_x k = 0 \tag{15a}$$

with the same equation for ω.

Hence, we have a nonlinear hyperbolic equation for the wavenumber (or frequency) of the wavetrain in a dispersive medium. The solution has the form of a simple (Riemann) wave well known in gas dynamics:

$$k = F[x - c_g(k)t] \tag{16}$$

or

$$x - c_g(k)t = \Phi(k),$$

where F and Φ are arbitrary reciprocally inverse functions defined by the initial conditions.

The wave amplitude can be found from the energy balance equation

$$\partial_t E + \partial_x(c_g E) = 0, \tag{17}$$

where E is the density of wave energy and $c_g E$ is the density of energy flux. Since c_g is already known from (16), Equation (17) is linear with respect to E and can be solved to give [51]

$$E = \frac{Q(\Phi)\partial_x k}{d_\Phi k(\Phi)} = \frac{Q(\Phi)}{1 + t d_\Phi c_g} \tag{18}$$

where $Q(\Phi)$ is defined by the initial condition.

Note that the evolution of the Riemann wave (16) commonly results in its “breaking”, i.e. the intersection of group trajectories – “the space-time rays” (16) at which $\partial_x k$ and E have points of divergence [51]. The wave behavior at these points needs a more exact description.

Here, we shall consider another important question, how the wave behaves at large distances from the region of its initial localization, as $x, t \to \infty$. As it can be seen from (16), far from the initial localization region $c_g(k) \to x/t$ which, for the known dispersion equation, immediately gives the dependence $k(x, t)$. As regards to the energy, it has the following asymptotic form

$$E = \frac{Q_1(\Phi)}{t} = \frac{S(k)}{t d_k c_g}, \tag{19}$$

where $Q_1 = Q(\Phi)/d_\Phi c_g$ and $S(k)$ is the spectral energy density of the initial wavetrain. Hence, due to dispersion, the wave packet broadening occurs in such a way that different wave groups are placed separately on the x-axis (a "space-time prism").

The same results can also be obtained by the well-known method of stationary phase [74] from the exact solution to the linear initial-value problem. Let us now apply these asymptotic formulae to the dispersion relation (8). We shall consider three basic cases (refering now to waves propagating in the positive x-direction).

2.2.1. *Nonrotating fluid,* $(f = 0)$

Here $\omega \approx c_0 k - \beta k^3$ (as in 7), so that $c_{ph} = c_0 - \beta k^2$, and $c_g = c_0 - 3\beta k^2$. Thus for large x, t we have (for $\beta > 0$

$$k = \sqrt{\frac{1}{3\beta}\left(c_0 - \frac{x}{t}\right)}, \quad x < c_0 t \tag{20}$$

The wave energy evolution is given by

$$E = \frac{S(x/t)}{-2t\sqrt{3\beta(c_0 - x/t)}}, \tag{21}$$

where S is defined by an initial condition. The formula (21) shows, first, that the intensity of a given group decays as t^{-1}, i.e. the wave amplitude decays as $t^{-1/2}$ for any given wave group. This result agrees with the self-similar solution of the linearized KdV equation [59], which can be presented in the form

$$\eta \sim \frac{1}{(\beta t)^{1/3}} F[\frac{x - c_0 t}{(\beta t)^{1/3}}], \tag{22}$$

where $\eta(x, t)$ is the elevation of sea level for surface waves or the displacement of an isopycne for the case of internal waves. The function $F(x)$ for the linearized KdV equation can be easily found and it is nothing more than Airy function [37, 2]. The far-field asymptotic form of this solution corresponds to a modulated wave with an amplitude decreasing as $t^{-1/2}[1 - x/(c_0 t)]^{-1/4}$.

Secondly, the wave amplitude in (21) formally grows to infinity at the wave front, i.e. as $x \to c_0 t$. This follows from the fact that $k \to 0$ at the front, and the wave becomes nondispersive so that its energy remains concentrated. This result is, however, not exact; of course, in the full solution the frontal part remains finite and corresponds again to the Airy function. To be more precise one can easily obtain from the exact solution of the linearized KdV equations a self-similar first term in the asymptotic expansion for large t (see, e.g., [37, 2] for details) for initially localized perturbations

$$\eta(x,t) \sim \frac{M}{(3\beta t)^{1/3}} \mathrm{Ai}(z), \tag{23}$$

where the argument of the Airy function $z = (x - c_0 t)/(3\beta t)^{1/3}$, while the “mass” of a wave

$$M \equiv \int_{-\infty}^{\infty} \eta(x,0)\, dx \tag{24}$$

is a constant.

Let us now make some estimates. For surface waves, suppose the initial impulse to be generated as a tsunami wave, so that it has a characteristic length of about $100\,km$ and thus its spectrum spans from $k = 0$ to $k_{max} \approx 6 \cdot 10^{-5}\, m^{-1}$. In an ocean of $4\,km$ depth, $c_0 = 200\, m/s$, and $c_g(k_{max})$ is only about $6\, m/s$ less. Thus, to spread from 100 to 200-km length , the pulse must travel about $7500\,km$, which is of the order of an ocean basin size (however we can make only a rough estimate, for our formulae are applicable only for large spreading). Hence, only very moderate dispersive deformations of tsunami waves in the deep open ocean can be expected. However, as the tsunami propagates into shallow water, it will slow down and dispersive effects can become more relevant. Much more significant effects can be expected for internal waves.

One of the fundamental questions to be answered by this linear theory is whether an initially linear perturbation becomes essentially nonlinear, and if yes, when, and on what features of the initial perturbation does this depend. The general answer to this question can be found by calculating the neglected nonlinear terms on the basis of these *linear* solutions. For this particular problem taking into account nonlinearity results in an eventual transformation of an initial localized disturbance into solitary waves provided the initial disturbance has positive “mass” M, defined by (24). In this generic situation of nonzero “mass” the characteristic time t_N can be easily estimated using (23). Indeed, if we start from an initial perturbation for which the Ursell parameter (the ratio of nonlinear to dispersive terms) is small, $Ur_0 \sim \eta_0 L_0^2 \ll 1$, where η_0 is the initial amplitude scale, and L_0 is the initial characteristic length scale, then we obtain with the help of (23), that $Ur(t) \sim t^{-1/3} \cdot t^{2/3} \sim t^{1/3}$, i.e. it grows with time. Hence nonlinear effects eventually become as significant as dispersive effects. For KdV solitary waves, the Ursell parameter is of order unity (see, e.g., [2, 37]), and the characteristic time for the manifestation of nonlinear effects can be estimated to be [1, 59, 45]

$$t_N \sim \frac{C}{(M\alpha\beta)^3}, \tag{25}$$

where α is a nonlinear coefficient (see below in § 3.2) and C is a numerical constant, depending *inter alia*, on the Ursell parameter.

2.2.2. *Rotation without high-frequency dispersion,* $(\beta = 0)$
This type of wave motion is commonly called Poincarè waves [42]. In this case

$$c_g = \frac{c_0}{\sqrt{1 + f^2/(c_0 k)^2}} \tag{26}$$

The asymptotic formula for k as $x \to \infty$ now has the form

$$k = \frac{f}{c_0\sqrt{c_0^2 t^2/x^2 - 1}} \tag{27}$$

and, for the energy

$$E = \frac{S(k) f}{c_0^2 t[1 - x^2/(c_0 t)^2]^{3/2}} \tag{28}$$

for $x < c_0 t$. In this case high-frequency groups propagate in front of the wavetrain, and the energy is again concentrated near the front, but now because of the absence of high-frequency dispersion. The long-wave part of the train disperses around the initial point, without any systematic motion.

Of interest in this case is that discontinuities are not dispersed. As an example, Gill [19] has calculated the evolution of a stepwise initial perturbation and observed the development of oscillations behind the discontinuity corresponding to the solutions described above, including the decay of the horizontal velocity as $t^{-1/2}$ at the pulse trail. He also refers to observations made for internal Poincarè waves in Lake Ontario where processes corresponding in some aspects to this stepwise front evolution were seen and compared (by Simons [69]) with the theoretical calculations. Note, however, that the observed processes obviously included nonlinear effects as well.

Estimates for a pulse of 100 *km* length mentioned above give extremely weak dispersion broadening (less than 1 cm/s) so that the dispersive effects due to rotation are hardly important for deep-ocean tsunamis on these scales. However if we take the initial pulse to be longer, of order 10^3 *km* (although rarely tsunamis of such scales do occur [58]) we obtain manifestations of dispersive effects at scales of

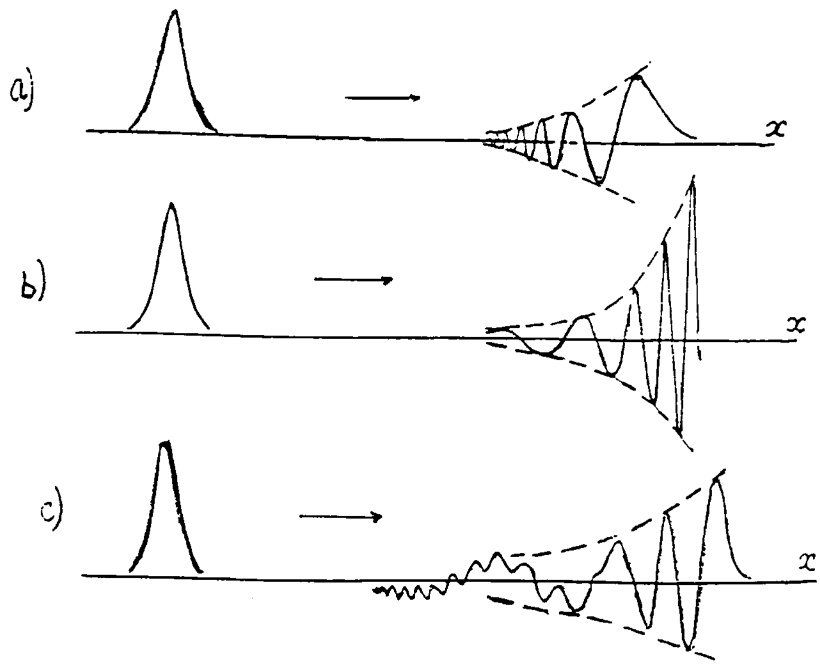

Figure 2. Schematic pictures of linear wave evolution due to dispersion in three different cases: (a) $f = 0,\ \beta \neq 0$; (b) $\beta = 0,\ f \neq 0$; (c) $f \neq 0,\ \beta \neq 0$.

about 10^4 *km*. Thus one may expect some importance of rotation, say, for tsunamis crossing the Pacific ocean.

The situation with the internal waves is again quite different, a pulse of realistic length, say $10 - 10^2$ *km*, spreads at its own scale rather quickly. We shall discuss internal wave dispersion in more detail in context of the nonlinear propagation.

2.2.3. *General case,* $(f \neq 0,\ \beta \neq 0)$

This case is more complicated. To simplify the problem, we again consider weak dispersion when (9) is valid. Here, the asymptotic behavior of k is defined by the relation $c_g = x/t$.

It is easy to see that the group velocity for surface waves has a maximum at $k = k_{in} = \sqrt{f/(c_0 H)}$, which is $(c_g)_{max} = c_0 - Hf$. A similar result holds for internal waves, the only essential difference being the presence of different numerical factors. This means that groups with this value of k will be at the wave front, while for the smaller c_g there exist two possible values of k. Hence, we should observe two-frequency quasiperiodic perturbations behind the wave front.

Schematic pictures of wave evolution in the above listed three cases are given in Figure 2.

2.2.4. *Two-layer fluid with a deep lower layer*

Finally, we consider the important special case of waves on the interface between a thin layer lying over a deep lower layer. For the case when this lower layer is infinitely deep, the dispersion relation is given by (13). For a progressive wave with weak dispersion we have

$$\omega = c_0 k + \frac{f^2}{2c_0 k} - \frac{\rho_2 c_0 H_1}{2\rho_1} k|k|. \tag{29}$$

Here we also have an inflection point at $k_{in} = (\rho_1 f^2/\rho_2 c_0^2 H_1)^{1/3}$. For internal waves we can take, say, $(\rho_2 - \rho_1)/\rho_1 = 10^{-3}$, $H_1 = 50\,m$; then $c_0 \simeq 0.7\,m/s$, and $k_{in} \approx 7.5 \cdot 10^{-4}\,m^{-1}$ which corresponds to the wavelength of about $8.5\,km$. Around this spatial scale, we can expect a two-frequency regime to form.

Note that if we take into account only the high-frequency dispersion (i.e. $k \gg k_{in}$), the parameter $d_k c_g$ does not depend on the wavenumber at all. That means that, according to (19), any part of the wave spectrum in which $S(k)$ =const will spread homogeneously in space so that E does not depend on x in this interval. If we use again the self-similar solution with nonzeromass for this case we obtain, similarly to (22),

$$\eta \sim (c_0 H_1 t)^{-1/2} F[(x - c_0 t)/(c_0 H_1 t)^{1/2}]. \tag{30}$$

But now the Ursell parameter $Ur = \alpha\eta_0 L_0/\beta$ does not depend on time. Hence any linear perturbation remains linear for all times.

3. Structure and Dynamics of Nonlinear Waves

Now we shall pass to the main topic of this review, that is, nonlinear effects for rotating fluids in the range of intermediate length scales defined above.

3.1. Governing equations

The waves to be considered are typically of a long wavelength, and the modal representation of the field is adequate. It then follows that the basic equations (1) can be reduced to a system of Boussinesq-type equations corresponding to a single-layered fluid of moderate depth (see, e.g., [56, 19]).

For surface waves the derivation of such a system is straightforward and well-known. When the rotation term is added to the Boussinesq equations, we get [56, 19]

$$\begin{aligned} \partial_t \eta + \nabla_\perp[(H+\eta)\mathbf{q}] &= 0, \\ \partial_t \mathbf{q} + (\mathbf{q}\cdot\nabla_\perp)\mathbf{q} + 2[\Omega \times \mathbf{q}] + g\nabla_\perp\eta + \tfrac{H}{3}\nabla_\perp\partial_t^2\eta &= 0, \end{aligned} \tag{31}$$

where $\mathbf{q}$ is the horizontal velocity vector, and η is the surface elevation. Note that the last term in the second equation which represents the leading-order non-hydrostatic linear effects, is supposed to be small compared to the other linear terms. Although, in general the nonlinear terms in (31) are formally exact in the hydrostatic limit, we shall suppose here that the nonlinear terms in both equations are small values of the same order as the dispersion term. Further, while the rotational term can in general be comparable to the other linear terms, we shall suppose it small too, so that the high-frequency and low-frequency dispersion effects may be comparable.

For internal waves equations analogous to (31) may also be derived although in a more complicated way and only for *a priori* weak nonlinearity [53]. The idea of the derivation is the following. Starting with the basic equations (1) for a stratified fluid, we separate the velocity into its horizontal component, $\mathbf{q}$, and vertical component, w, and then rewrite the equations in the following form

$$\begin{aligned}
\rho_0\partial_t\mathbf{q} + \nabla_\perp p + 2\rho_0[\Omega\times\mathbf{q}] &= -\rho\partial_t\mathbf{q} - 2\rho[\Omega\times\mathbf{q}] - \\
&\quad \rho_0(\mathbf{q}\nabla_\perp)\mathbf{q} - \rho_0 w\partial_z\mathbf{q} + \ldots, \\
(\nabla_\perp\cdot\mathbf{q}) + \partial_z w &= 0, \\
\partial_t\rho + w d_z\rho_0 &= -(\mathbf{q}\cdot\nabla_\perp)\rho - w\partial_z\rho, \\
\partial_z p + g\rho &= -\rho_0\partial_t w + \ldots.
\end{aligned} \tag{32}$$

Here the omitted terms are either of higher order in nonlinearity or in dispersion than is needed in the sequel. Then for small nonlinearity and dispersion a perturbation theory can be used. A solution is sought by an expansion in the form of the modal functions,

$$w = \sum W_m(z)w_m(x, y, t), \qquad \mathbf{q} = H\sum d_z W_m\mathbf{q}_m(x, y, t), \tag{33}$$

and similarly for ρ and p. Here W_m are the modal functions in the long wave limit, and in the absence of rotation. They are defined by (5) in this limit with $f = 0$.

To obtain the Boussinesq-type equations, it is useful to introduce a variable η that characterizes the vertical displacement of isopycnal surfaces. Along such a surface, $w = \partial_t\eta + (\mathbf{q}\nabla_\perp)\eta$, and at $z = \text{const}$ we have $w \approx \partial_t\eta + \nabla_\perp(\eta\mathbf{q})$. Using the modal representation of η:

$$\eta = \sum W_m(z)\eta_m(x, y, t), \tag{34}$$

substituting this into (32), and utilizing the orthogonality of the set $\{W_m(z)\}$ by multiplying each equation by W or $\partial_z W$ and integrating over z from $-H$ to 0, we get, omitting the subscript m,

$$\partial_t \eta + H(\nabla_\perp \cdot \mathbf{q}) + \frac{s+s'}{2}\nabla_\perp(\eta\mathbf{q}) = 0,$$

$$\partial_t \mathbf{q} + \frac{c_0^2}{H}\nabla_\perp \eta + 2[\Omega \times \mathbf{q}] + s[\mathbf{q} \times \nabla_\perp \mathbf{q}] + \tag{35}$$

$$\frac{1}{2}(s-s')(c_0^2\nabla_\perp\eta - \partial_t \eta \mathbf{q}) + \sigma H \nabla_\perp \partial_t^2 \eta = 0.$$

Here,

$$s = \frac{H\int_{-H}^{0} \rho_0 (d_z W)^3\, dz}{\int_{-H}^{0} \rho_0 (d_z W)^2\, dz}, \qquad \sigma = \frac{\int_{-H}^{0} \rho_0 W^2\, dz}{H^2 \int_{-H}^{0} \rho_0 (d_z W)^2\, dz} \tag{36}$$

are the parameters of nonlinearity and high-frequency dispersion respectively, while

$$s' = \frac{H\int_{-H}^{0} \rho_0 d_z[W(d_z W)^2]\, dz}{2\int_{-H}^{0} \rho_0 (d_z W)^2\, dz}.$$

The value of s' is zero in the Boussinesq approximation. Note, that the parameters s and s' depend on a calibration of the eigenfunctions W, thus determining the respective amplitudes for η and $\mathbf{q}$. More specifically, the latter correspond to the level where W is chosen to equal unity.

As an example, let us give the values of the nonlinearity and dispersion parameters in the Boussinesq approximation (which is practically always sufficient for the ocean) in the case of a two-layer fluid:

$$s = \frac{h_1^2 - h_2^2}{h_1 h_2}; \qquad \sigma = \frac{h_1 h_2}{3H^2},$$

and W is normalized to be unity at the interface.

Finally, after a simple substitutions of variables

$$s\eta \to \eta, \qquad s\mathbf{q} - \frac{1}{2}s(s-s')\eta\mathbf{q} \to \mathbf{q}, \tag{37}$$

this system reduces to the standard form (31),

$$\partial_t \eta + \nabla_\perp[(H+\eta)\mathbf{q}] = 0,$$

$$\partial_t \mathbf{q} + (\mathbf{q}\cdot\nabla_\perp)\mathbf{q} + 2[\Omega \times \mathbf{q}] + \frac{c_0^2}{H}\nabla_\perp\eta + \sigma H \nabla_\perp \partial_t^2 \eta = 0. \tag{38}$$

The peculiarities of the case of the internal waves are associated with the dependence of the equation parameters on the mode number, and, with the fact that the nonlinearity parameter s may have different values and even signs (while the dispersion parameter σ is always positive). Of course, the characteristic values of the corresponding parameters are significantly different for all these modes. With this restriction, the systems (31) and (38) can be treated by the same methods, considering the surface wave as just another mode of the gravity wave spectrum.

The single-mode equations (38) are valid, in this approximation, provided there are no special resonance (synchronism) conditions fulfilled which might provide cumulative energy exchange between different modes. In the latter case, coupled Boussinesq-type equations can be used [53].

3.2. NONROTATING FLUID, $(f = 0)$

Let us first recall some known results concerning waves without rotational effects, i.e., $\Omega = 0$. For a progressive wave propagating in the x-direction, the Boussinesq equations yield the Korteweg – de Vries (KdV) equation

$$\partial_t \eta + \alpha\eta\partial_\xi \eta + \beta\partial_\xi^3 \eta = 0, \tag{39}$$

where the variables $\xi = x - c_0 t$ and t are introduced instead of x and t. The values of the parameters for the surface waves are $c_0 = \sqrt{gH}$, $\alpha = 3c_0/2H$, $\beta = c_0 H^2/6$, while for internal waves: $\alpha = 3sc_0/2H$, $\beta = c_0\sigma H^2/2$, and c_0 is determined as an eigenvalue of the boundary-value problem (5) for each mode.

There are many derivations of the KdV equation in the literature (see, for instance, the review article [28] for the case of internal waves). Here it is sufficient to note that the linear dispersion relation for the KdV equation (39) exactly coincides with the linear dispersion relation (7) for unidirectional linear waves, while the nonlinear coefficient α is readily found from (31) or (35) with the Coriolis and dispersive terms omitted, by finding the weakly nonlinear expansion of that Riemann invariant which is propagating to the right.

Equation (39) is known to be completely integrable (see, e.g. [2]), and we give here only the solitary wave solution (a soliton):

$$\eta = \eta_0 \operatorname{sech}^2(\zeta/\Delta), \tag{40}$$

where $\zeta = \xi - Vt$, and the velocity V (in the reference frame moving with the speed c_0) and the characteristic length of the soliton Δ are related to its amplitude η_0 by

$$V = \frac{\alpha\eta_0}{3}, \qquad \Delta = \sqrt{\frac{12\beta}{\alpha\eta_0}} \tag{41}$$

An important special case is that when the length of the internal wave exceeds the thickness only of an upper layer lying over a deep nonstratified lower layer so that we have a condition of "shallow water" for the upper layer and of "deep water " for the lower. The dispersion relation for the internal wave modes in this case was discussed above [see (13), or (29)]. The nonlinear evolution equation corresponding to this dispersion relation leads to another completely integrable equation, the Benjamin – Ono (BO) Equation (see, e.g., [2]),

$$\partial_t \eta + \alpha\eta\partial_\xi \eta + \frac{\beta}{\pi}\partial_\xi^2 \wp\int_{-\infty}^{\infty} \frac{\eta(t,\xi')}{\xi-\xi'}\,d\xi' = 0, \tag{42}$$

where $\wp$ means that the principal value of the integral is to be taken. Its solitary wave solutions are the "algebraic" solitons,

$$\eta = \frac{\eta_0}{1+\zeta^2/\Delta^2} \tag{43}$$

with the parameters

$$V = \frac{\alpha\eta_0}{4}, \qquad \Delta = \frac{4\beta}{\alpha\eta_0} \tag{44}$$

There also exists a more general nonlinear evolution equation, the intermediate long wave equation, suggested by Joseph [34] and Kubota et al. [40] which unifies both the KdV and BO equations as two extreme cases of the single integrable equation.

3.3. ROTATION-MODIFIED KDV EQUATION

Let us return now to the rotational case. Equations (31) and (38) contain two types of dispersive terms, low-frequency dispersion (with Ω) and high-frequency dispersion (with H). If both are small enough (together with the nonlinear terms), a generalization of the KdV-equation can be easily obtained for a progressive wave in which all small terms appear additively,

$$\partial_\xi(\partial_t\eta + \alpha\eta\partial_\xi\eta + \beta\partial_\xi^3\eta) = \delta\eta, \tag{45}$$

where $\delta = f^2/(2c_0) \geq 0$.

The derivation of Equation (45) is analogous to that for the KdV Equation (39). Indeed, since here the Coriolis term is assumed to be of the same order as the high-frequency dispersive term, it is sufficient to note that the linear dispersion relation for (45) exactly coincides with the linear dispersion relation (9) for unidirectional linear waves when the Coriolis term is included.

Equation (45) (which we will call the rKdV-equation) was first derived by Ostrovsky for internal waves [53]. It is, of course, valid for surface waves as well. Later it was generalized and analyzed by different authors (see, e.g. [43, 64, 27, 38, 55, 18] and others). A slightly more general equation, which we will discuss later, was derived by Odulo [50] and Shrira [67, 68]. An equation similar to (45) but without high-frequency dispersion ($\beta = 0$) was derived by Muzylev [47] for inertial-gravity waves in the equatorial ocean. Besides, exactly the same equation as (45) was shown to describe weakly nonlinear long waves of arbitrary physical origin in random media [6] and surface water waves over periodic or random bottom irregularities [5].

Let us discuss first some general properties of this equation. When considering the localized solutions of (45), some important integral constraints can be obtained. One of them is the "zero-mass" condition,

$$M = \int_{-\infty}^{\infty} \eta \, d\xi = 0. \tag{46}$$

Another is the energy integral,

$$I = \frac{1}{2} \int_{-\infty}^{\infty} \eta^2 d\xi = \text{const}. \tag{47}$$

A third is the Hamiltonian,

$$\mathcal{H} = \frac{1}{2} \int_{-\infty}^{\infty} \left[\beta \left(\partial_\xi \eta\right)^2 - \frac{\alpha}{3}\eta^3 - \delta(\partial_\xi \varphi)^2 \right] d\xi = \text{const}. \tag{48}$$

where φ is related to η by $\eta = \partial_\xi^2 \varphi$ so that the Equation (45) can be presented in the form

$$\partial_t \eta = \partial_\xi \frac{\delta \mathcal{H}}{\delta \eta}. \tag{49}$$

Other integrals have been first discovered by Benilov [5]. The simplest of them is

$$\int_{-\infty}^{\infty} \xi \eta \, d\xi = 0. \tag{50}$$

All these integrals are definite so that they play a role of constraints rather than conservation laws. These integrals are valid for localized perturbations which vanish sufficiently fast when $|\xi| \to \infty$, but the first three are also valid for any periodic solution if the integrals are taken over the period.

The existence of these integrals does not, of course, imply complete integrability and as far as it is presently known, the rKdV Equation (45) is not integrable. Note that these integrals do not nessessarily remain valid for the more general Boussinesq-type equations written above; for example, (46) holds, in the latter case, for the class of steady-state waves only.

3.3.1. *Steady waves*

Low-frequency waves. Not only is the rKdV equation most likely non-integrable, to this point there are no known explicit analytical solutions. A relatively complete analysis has been performed only for steady-state waves of the form $\eta = \eta(\zeta)$, where again $\zeta = \xi - Vt$ with constant V. In this case, (45) reduces to the fourth-order ordinary differential equation,

$$d_\zeta^2 \left[\beta d_\zeta^2 \eta + \alpha\eta^2/2 - V\eta\right] = \delta\eta. \tag{51}$$

Let us start from the case when the rotational (low-frequency) dispersion dominates so that one can let $\beta \simeq 0$. Thus, we obtain a second-order equation,

$$d_\zeta^2 \left[\alpha\eta^2/2 - V\eta\right] = \delta\eta. \tag{52}$$

This equation was investigated in [53, 54]. Its phase portraits and some wave profiles for fast ($V > 0$ so that the wave propagates faster than a linear long wave), and slow ($V < 0$) waves are shown in Figs. 3 and 4. It is seen that the periodic fast waves are possible, but slow periodic waves do not occur. Among the fast periodic waves one can distinguish small amplitude quasisinusoidal waves, nonlinear waves with smooth profile, and limiting-form waves which correspond to the separatrix (heavy line) in Figure 3, case (a), and have the parabolic form,

$$\eta = \frac{1}{2\alpha}\left(\frac{\delta\zeta^2}{3} - V\right). \tag{53}$$

It is pertinent to note that the (unbounded) parabolic arc (53) is in fact an exact solution of the full Equation (51) for all β and V.

If a periodic wave is composed of pieces of such parabolas (see Figure 4), then the wavelength must be equal to $6\sqrt{V/\delta}$, to secure the zero-mass condition (46) (note that the wavelength of an infinitesimal-amplitude sinusoidal wave is equal to $2\pi\sqrt{V/\delta}$).

In contrast, slow solutions cannot be both limited and smooth at the same time. Indeed there is a singular solitary wave solution which has a speed $V < 0$, but has a singularity (infinite derivatives) at the wave crest, for which $\eta_m - \eta$ varies locally as $\sqrt{\zeta}$, where $\eta_m = V/\alpha$. This singular solution corresponds to the orbit in Figure 3, case (b), which connects the saddle point to plus/minus infinity at $\eta = \eta_m$. But we note that the presence of the singularity enables this solution to violate the zero-mass condition (46), since the corresponding integrals in deriving this condition are not specified. Of course, the presence of such a singularity points to the necessity to include the high-frequency dispersive term, that is, to allow $\beta \neq 0$. We shall show below that in this case if $\beta > 0$ (the usual case in the ocean or atmosphere) then there is no steady solitary wave solution of (51). But if $\beta < 0$ (which cannot be realized for oceanic or atmospheric internal waves), then slow, nonsingular solitary

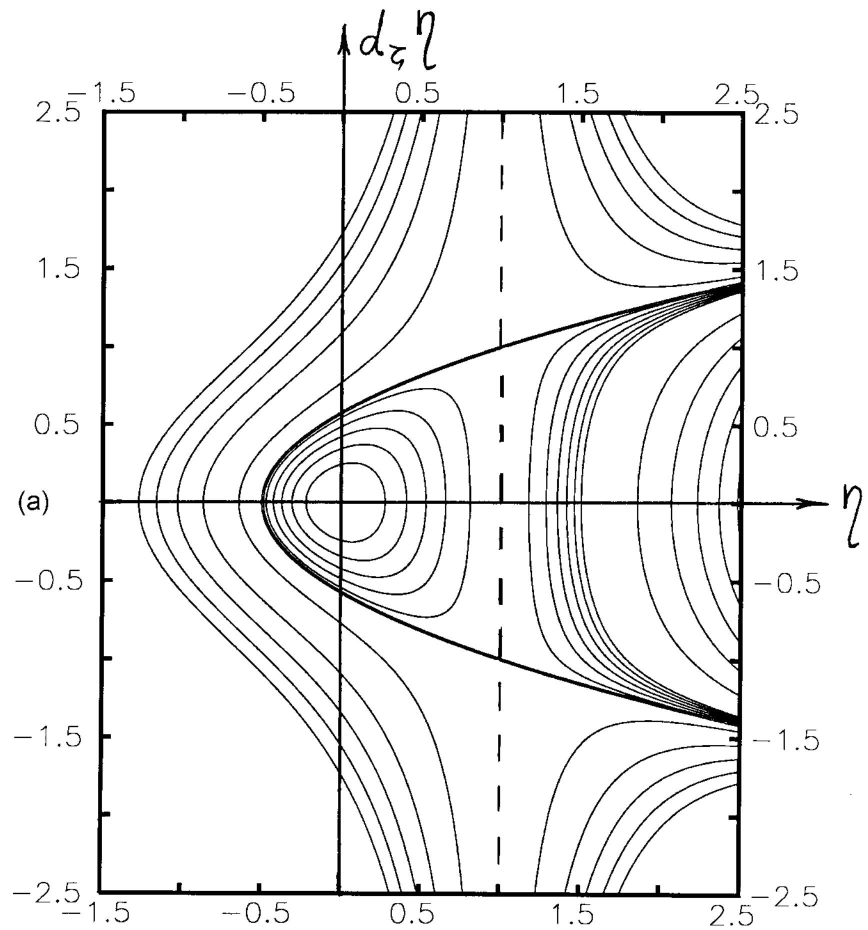

Figure 3. Phase portraits of the Equation (52) for fast waves (a) and slow waves (b).

waves, with smooth profiles and zero total mass, may exist (see below in the next sub-section). *

It is worth noting here one important circumstance. Let us look again at the unsteady Equation (45) with $\beta = 0$. If the right-hand side of this equation is equal to zero, it describes a simple (Riemann) wave. It is well known that the solutions of this equation for a smooth initial perturbation generally exist only for a finite period of time [41]. Due to nonlinear effects, the solutions become multivalued, which usually means the formation of discontinuities (shocks). From the spectral viewpoint, this process corresponds to the unlimited generation of

* In a quite different physical context where the high-frequency dispersion is identically zero (see, e.g. [47, 6, 66]), a different class (known as "compactons" [65]) of confined, but singular slow solutions may be of interest. Also the total "mass" of such solutions is likewise not equal to zero. Indeed, such "compacton-like" solutions can also be found for Equation (52) by using other "slow" orbits in Figure 3, case (b).

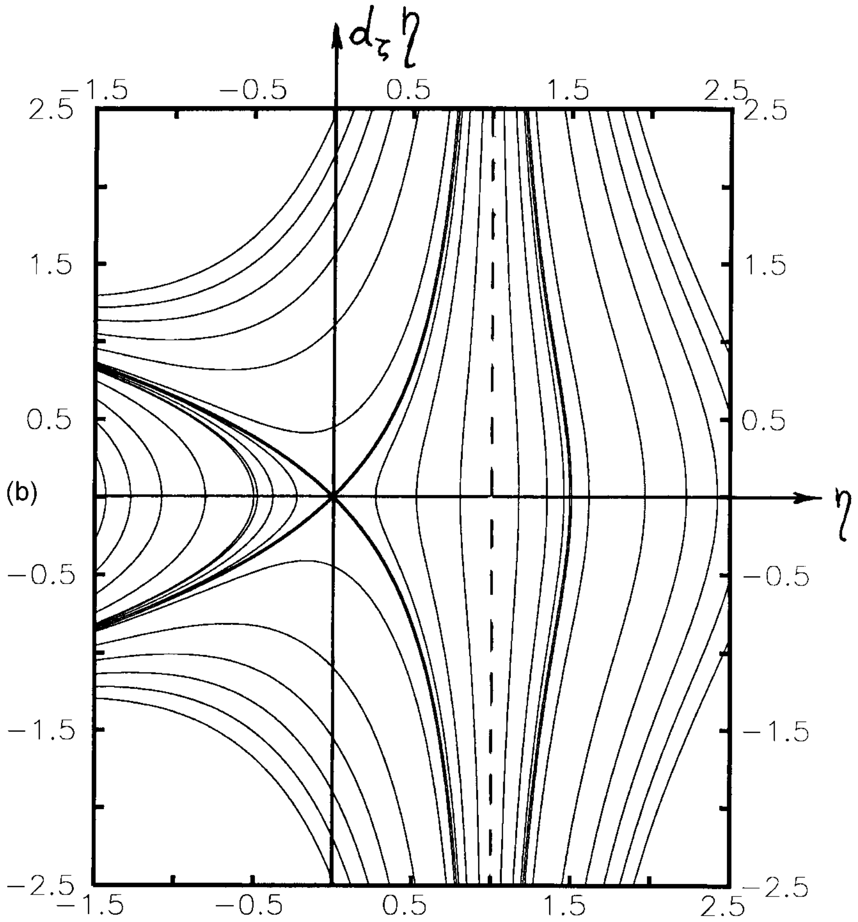

Figure 3.

higher harmonics. Usually *high-frequency* dispersion [e.g. term with $\beta \neq 0$ in the Eq. (45)] or dissipation act to impede the generation of such harmonics, and lead to the formation of steady-state waves. But if we have just *low-frequency* dispersion [$\delta \neq 0,\ \beta = 0$ in (45)], the existence of discontinuities is evidently possible. Still, the question remains; can such a dispersion prevent shock formation and instead create a steady-state wave? The above results indicate that there exists a class of smooth steady-state solutions of (45) with $\beta = 0$. The question about the stability of these solutions is still open. On the other hand, it can be shown (see [30]) that for a wide class of initial conditions for which the condition $\partial_\xi^2 \eta = 0$ is satisfied at the same point where $\eta = 0$, and $\partial_\xi \eta < 0$ at this point, a shock is always formed and occurs first for positive values of η. This class includes all smooth odd functions, such as a sinusoidal initial conditon.

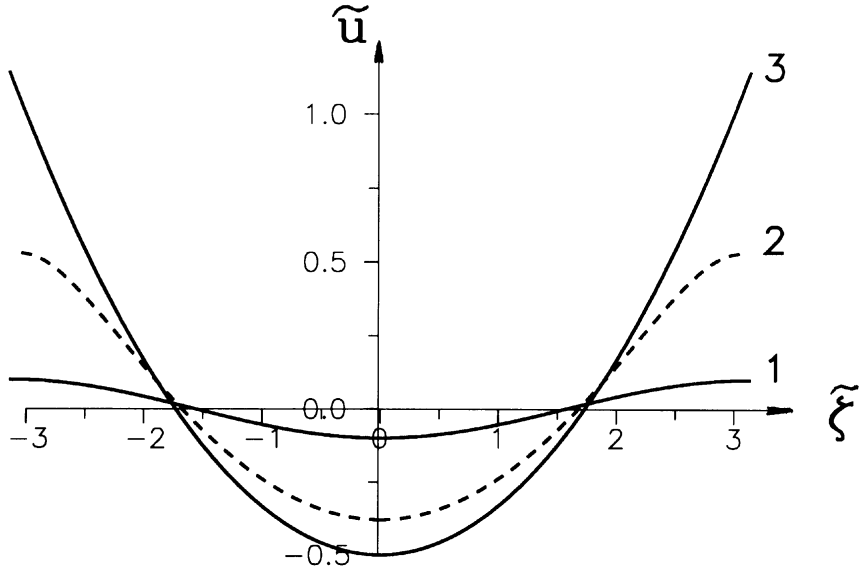

Figure 4. Periodic waves in a rotating fluid without high-frequency dispersion in dimensionless variables $\widetilde{\zeta} = \zeta(\pm\delta/V)^{1/2}$, $\widetilde{u} = \alpha u/V$: from a quasi-sinusoidal wave (1) through to the sequence of parabolic arcs (3).

Both dispersions. Returning to Equation (51) with $\beta \neq 0$, we first note that it is conservative and has a first (energy) integral,

$$\left[\partial_\zeta\left(\beta\partial_\zeta^2\eta + \frac{\alpha}{2}\eta^2 - V\eta\right)\right]^2 - \delta\left[2\beta\eta\partial_\zeta^2\eta - \beta(\partial_\zeta\eta)^2 + \frac{2}{3}\alpha\eta^3 - V\eta^2\right] = 2C. \quad (54)$$

By introducing a new variable p_1, so that $\eta = \partial_\zeta p_1$, (50) can be presented as a Hamiltonian system of two degrees of freedom with the Hamiltonian

$$\mathcal{H} = \frac{1}{2\beta}\left[\delta p_1^2 + \beta p_2^2 + 2\beta q_1 q_2 - V q_2^2 - \frac{\alpha}{3}q_2^3\right]. \quad (55)$$

The corresponding set of canonical equations is,

$$\begin{aligned} \partial_\zeta q_1 &= \frac{\delta}{\beta}p_1; \qquad & \partial_\zeta p_1 &= -q_2; \\ \partial_\zeta q_2 &= p_2; \qquad & \partial_\zeta p_2 &= -q_1 + \frac{V}{\beta}q_2 + \frac{\alpha}{2\beta}q_2^2. \end{aligned} \quad (56)$$

This system is quasilinear and involves only one nonlinear term in the last equation. So far no additional independent integrals of motion are known for this system, and hence we can't conclude if this system is completely integrable. Moreover, no explicit analytical solutions have yet been obtained either for (51) or for the equivalent Hamiltonian system (56).

An important statement is that for $\beta > 0$ (which is typical of the ocean and atmosphere), Equation (51) does not have solitary wave solutions at all ("antisoliton theorem"). This fact was first established by Leonov [43]. A simple proof was given by Galkin and Stepanyants [13] based on the integral (54). We will not reproduce here the simple mathematical justification of the "antisoliton theorem" but give a visual physical interpretation. The dispersion relation (9) for $\beta > 0$ is such that real phase speeds $c = \omega/k$ exist for all real k. This means that for any possible solitary wave, there will exist a synchronous linear wave ("tail") which must eventually destroy the radiating solitary wave. The exception is the nonrotating (KdV) case because there are then no propagating linear waves with phase velocities greater than c_0, and the fast wave $(V > c_0)$ may exist without radiation. At the same time, for $\beta < 0$ strongly localized solitary wave solutions do exist (see below).

An attempt to classify possible types of solutions to Equation (51) was undertaken in [55] based on numerical calculations. It was discovered that among the wide class of periodic solutions, there is a rather interesting construction which can be considered as a combination of a periodic sequence of quasi-parabolic arcs matched at the crests by smooth narrow pulses, the form of which is close to the KdV solitary waves (40). From the physical point of view, such solutions are quite natural. Indeed, the afore-mentioned solution (53) is exact for the complete Equation (51) except in an infinitely narrow region near the crests, where two different parabolic arcs are matched. In this region, high-frequency dispersion predominates over low-frequency dispersion. Hence, we conclude that in this narrow region the solution of the Equation (51) must be close to a solution of the KdV equation. On the basis of these physical arguments and numerical simulations, an analytical description of such solutions was explored by a perturbation method in the paper [20]. We will present below the main results of this paper.

Combined periodic waves. The idea of the solution (in a somewhat more general form) is as follows. First of all it is convenient to reduce Equation (51) to the dimensionless form

$$d_\xi^2\left(-Vu + \frac{3}{2}u^2 - \frac{b}{4}d_\xi^2 u\right) = \frac{\varepsilon^2}{2}u, \tag{57}$$

where $b = \pm 1$ is a parameter which indicates the type of high-frequency dispersion, and ε is a dimensionless parameter responsible for the low-frequency dispersion.

We consider a KdV soliton of the form (40) placed on the crests of a periodic profile (sequence of parabolic arcs) with a wavelength λ large enough so that $\Delta/\lambda \ll 1$. The method is a matching of the outer solution (a parabola) and the inner solution (a soliton) in each period of the wave where the expansion parameter is $\varepsilon^2\Delta^2$. The solution is written as the asymptotic series,

$$\begin{aligned} u &= \frac{b}{9\Delta^2} - \frac{b}{\Delta^2}\,\mathrm{sech}^2(\frac{\xi}{\Delta}) + \varepsilon^2 u_1(\xi) + \ldots, \\ V &= -\frac{2b}{3\Delta^2} + \varepsilon^2 V_1 + \ldots \end{aligned} \tag{58}$$

Substituting this into (57) and solving the linearized equation for the perturbation u_1, we obtain that the latter is given in the far field, $|\xi| \to \infty$, by a quadratic polynomial and, hence, can be naturally matched to the outer exact parabolic solution. Taking into account the zero-mass condition (46), the full asymptotic solution in the wave period $|\xi| \le \lambda/2$ is

$$\begin{aligned} u &= \frac{b}{9\Delta^2} - \frac{b}{\Delta^2}\,\mathrm{sech}^2(\frac{\xi}{\Delta}) + \frac{\varepsilon^2}{36}\left[\left(\xi \mp \frac{\lambda}{2}\right)^2 - \frac{\lambda^2}{12}\right] + \ldots, \\ V &= -\frac{2b}{3\Delta^2} + \frac{\varepsilon^2\lambda^2}{72}, \end{aligned} \tag{59}$$

with $\lambda/\Delta = 18$.

It can be shown that such a solution is stable with respect to small displacements of the soliton from the equilibrium position at the crest of the parabolic arcs. It has to be noted, however, that such stable crest-positioned solitons are possible only for the piecewise-parabolic background*.

To confirm these analytical results some computations have been performed for the steady Equation (57) [20]. Figure 5 gives a comparison of the analytical solution (59) with the result of direct numerical computation for $b = -1,\ \varepsilon = 0.5$ and $\Delta = 1$. It is seen that even for comparatively large values of the rotation parameter the approximate theory gives a satisfactory description of the solution. Numerically it was found that these solutions exist and they are stable even for quite moderate values of ε, up to $\varepsilon = 0.7$ for negative dispersion ($b < 0$), but only up to $\varepsilon = 0.3$ for positive dispersion ($b > 0$).

In a conclusion of this subsection we note that analogous solutions for the combined periodic waves can be constructed in the case of rotational modified BO (rBO) equation [27]. The main idea is that matching of outer solution (sequence of parabolic arcs) must be made with algebraic solitons (42) on constant background, which is just equal to the maximum value of the outer periodic wave. Note that in fact this idea is rather universal and applicable to other equations of this type including above considered rKdV equation.

* We mention that stationary perodic solutions of Equation (51) in the form of a series, having another structure, were also found in [18]. Note that the single soliton solution also obtained in this paper for the case of negative dispersion, $b < 0$, is apparently an artifact because it contradicts to the aforementioned “antisoliton theorem”.

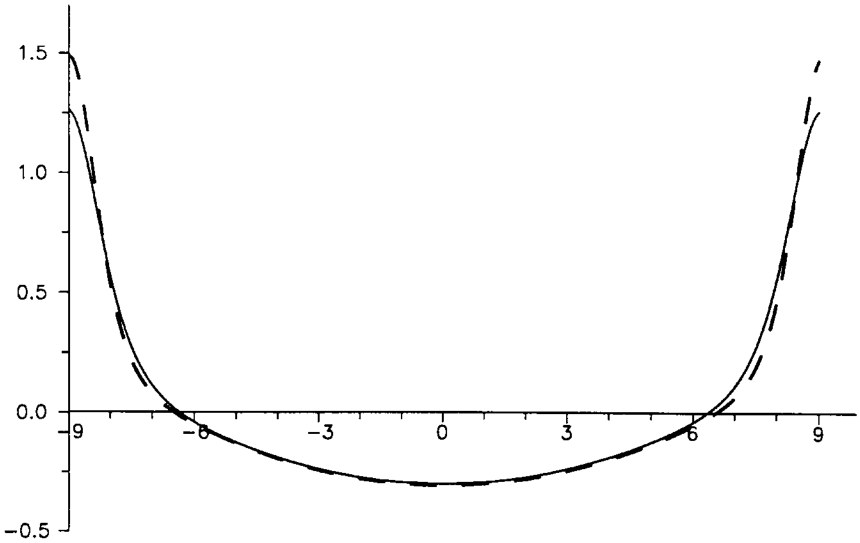

Figure 5. A comparison of the approximate analytical solution (59) of the Equation (57) (solid line) with the result of direct numerical computation (dashed line) for the same value of wave velocity (from [20]).

Periodic stationary solutions for the rBO equation which is analogous to the solution (59) for the rKdV equation have the form

$$\begin{aligned} u &= \frac{b}{9\Delta^2} - \frac{4b}{3\Delta^2 + \xi^2} + \frac{\varepsilon^2}{36}\left[\left(\xi \mp \frac{\lambda}{2}\right)^2 - \frac{\lambda^2}{12}\right] + \ldots, \\ V &= -\frac{2b}{3\Delta^2} + \frac{\varepsilon^2\lambda^2}{72}. \end{aligned} \tag{60}$$

The relationship between wave period and half-width of soliton is $\lambda/\Delta = 9\pi$.

3.3.2. *Nonstationary processes*

As regards nonstationary solutions of the rKdV equation (45), the first results were obtained numerically [55, 21]. Figure 6 shows the evolution of an initially KdV soliton for $\Delta = 1$ and $\varepsilon = 0.3$ according to the rKdV equation in the dimensionless form (61), so that the high- and low-frequency dispersions are of the same order. The soliton is destroyed rather quickly due to the rotation by radiating an oscillatory "tail". It is remarkable, however, that after some time the soliton-like profile (now on the low-frequency background) is almost completely restored! Such an incomplete but well pronounced recurrence (this process repeated again) is still to be explained, but it is clearly associated with the periodic boundary conditions used in the simulations.

Another numerical result is shown in Figure 7. It illustrates the propagation of a very narrow soliton ($\Delta \simeq 0.2$) of large amplitude on a weak parabolic background

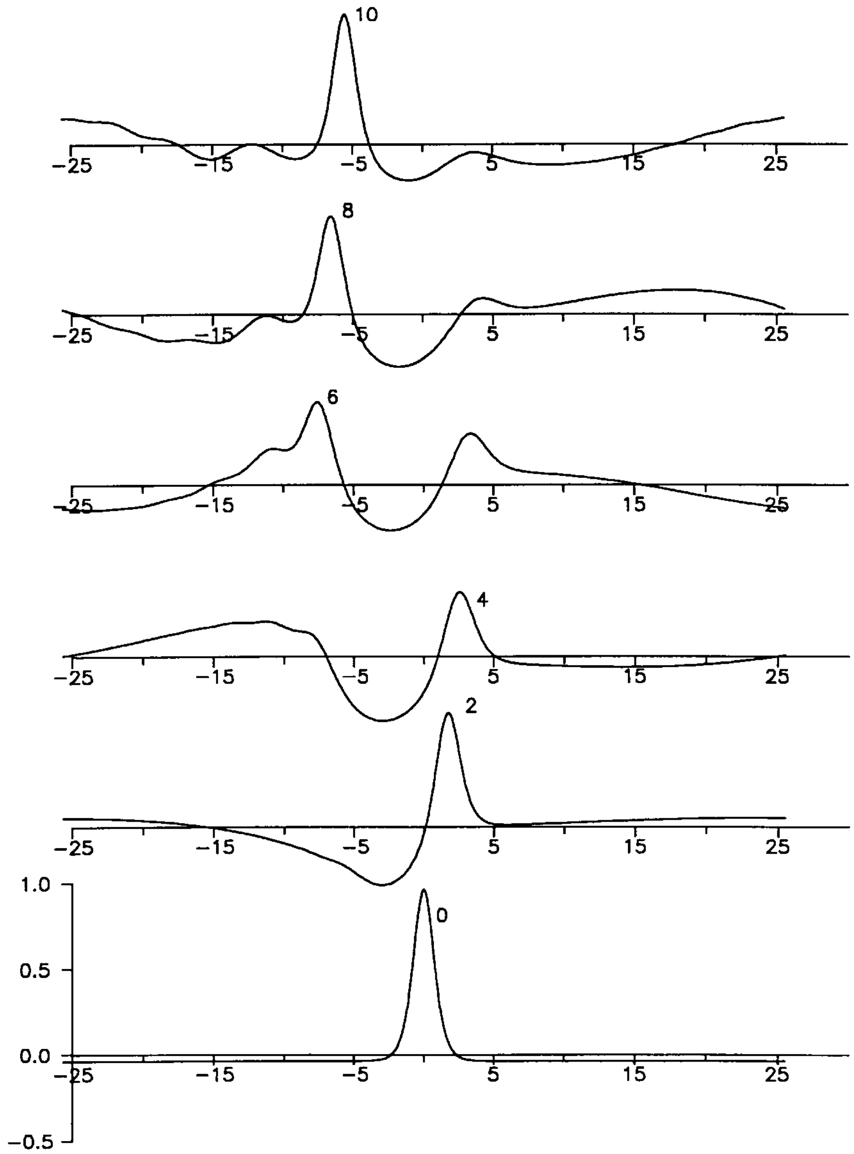

Figure 6. Evolution of a KdV soliton for $\Delta = 1$ and $\varepsilon = 0.3$ within the framework of the rKdV equation. Numbers near the wave crests indicate the time in dimensionless variables (from [21]).

with $\lambda = 32$ at $\varepsilon = 1$, so that $\lambda/\Delta \simeq 157$ (while in the stationary wave $\lambda/\Delta = 18$). As could be expected, such a soliton travels ahead through the background wave with a much larger velocity. However, it still "feels" the background in that its amplitude changes periodically depending on its position with respect to the phase of the long wave. The cause of this effect is the "parametric" energy exchange between the long wave and the soliton.

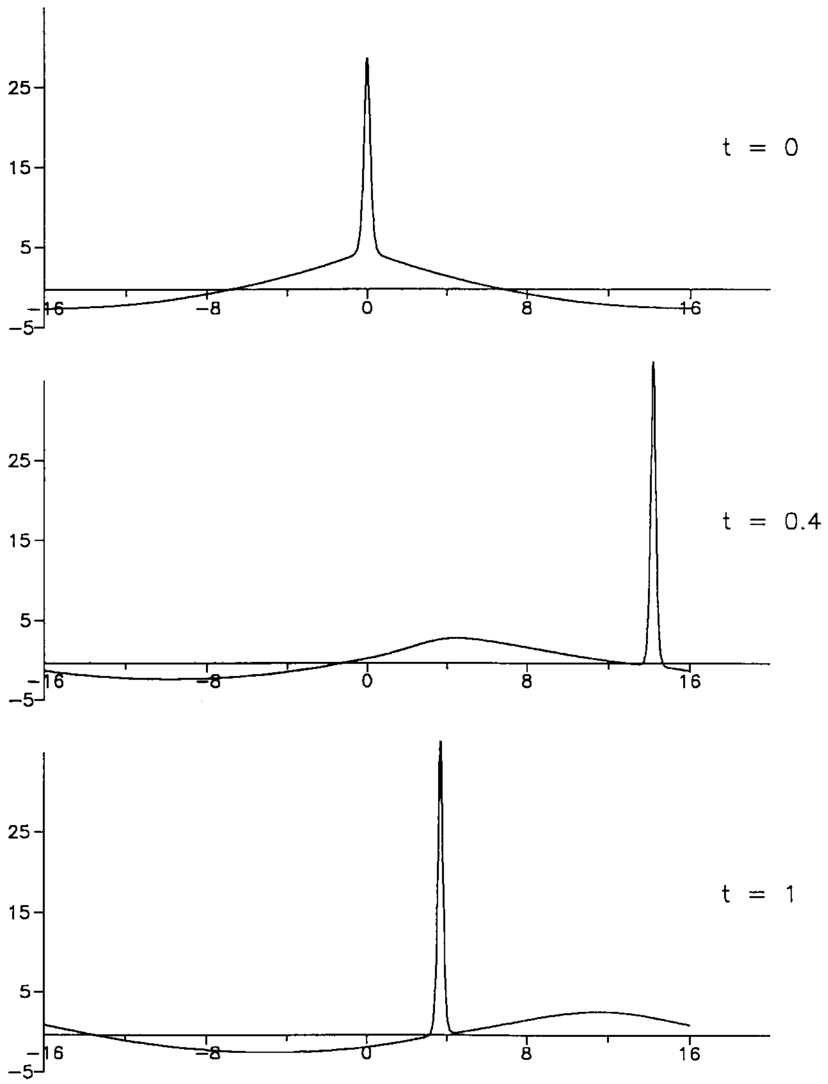

Figure 7. Adiabatic interaction of a strong KdV soliton with a weak parabolic wave within the framework of the rKdV equation (from [21]).

These processes can be described analytically by asymptotic methods [29]. Indeed, when ε is small we can consider, for example, the slow (adiabatic) decay of a KdV soliton due to the weak rotation effect resulting in the radiation of a long-wave “tail”. Starting from the Equation (45) in the dimensionless form for negative dispersion,

$$\partial_x \left(\partial_t u + 3u\partial_x u + \frac{1}{4}\partial_x^3 u \right) = \frac{\varepsilon^2}{2} u, \tag{61}$$

the solution in the vicinity of the soliton (inner expansion) is sought in the form

$$u = u_0(\theta, T) + \varepsilon^2 u_1(\theta) + \dots,$$
$$V = V_0 + \varepsilon^2 V_1 + \dots \tag{62}$$

Here $T = \varepsilon^2 t$, $\theta = x - \int_0^t V(T)\,dt$, $u_0 = A(T)\,\mathrm{sech}^{-2}\frac{\theta}{\Delta}$, $V_0 = A = \Delta^{-2}$. Then, a linearized equation for u_1 is obtained from (61),

$$d_\theta^2(-Vu_1 + 3u_0u_1 + \frac{1}{4}d_\theta^2 u_1) = -d_{T\theta}^2 u_0 + \frac{1}{2}u_0 \equiv d_\theta J(\theta), \tag{63}$$

for which the compatibility condition

$$\int_{-\infty}^{\infty} u_0(\theta) J(\theta)\,d\theta = 0 \tag{64}$$

securing the finiteness of u_1 must be fulfilled. After integration this gives a simple equation for the soliton amplitude A with the solution

$$A(T) = \frac{1}{4}(T_0 - T)^2, \tag{65}$$

where $T_0 = 2A_0^{1/2}$, $A_0 \equiv A(0)$. Thus, the soliton disappears in a finite extinction time T_0.

The radiation field (outer problem) has the form $u = \varepsilon U$, where

$$U(X, T) = B(X, T)\sin\varphi(X, T), \tag{66}$$

and $X = \varepsilon\theta$. Neglecting the high-frequency dispersion for it, we have a dispersion relation $\omega k = \varepsilon^2/2$ which defines the phase and group velocities of a radiating wave, $c_{ph} = -c_g = \varepsilon^2/2k^2$. Then, matching this with the soliton at $\theta = 0$, i.e. $x = x_s = \int_0^t V_0\,dt$, where $V_0 = c_{ph}$, we obtain that $B(0, T) = \varepsilon\sqrt{2}$ which is a constant. Then the dispersive propagation of the radiative tail behind the soliton can be easily described. Note that the radiation wavenumber is equal to $\varepsilon\Delta_0/\sqrt{2}$ in the beginning of the process and tends to infinity when $T \to T_0$, as it follows from the linear theory discussed above.

Let us now make some estimates of the extinction time T_0. Recalling the transformation of (45) to the dimensionless form (61), we readily find that the dimensional extinction time t_0 is given by,

$$t_0 = \frac{c_0}{f^2}\sqrt{\frac{\alpha\eta_0}{3\beta}} \tag{67}$$

For instance, consider an internal wave in the two-layer ocean with the pycnocline depth of order H_1. Note that, according to (36), we can estimate that $\alpha \approx nc_0/H_1, \quad \beta \approx c_0 H_1^2/n^2$ (n is the mode number). Then taking $c_0 = 1.5\,m/s,\ d = 500\,m,\ f = 10^{-4}\,s^{-1}$, and $\eta_0 = 50\,m$, we get that t_0 is about one day for the first mode and larger for the higher modes. This is in rough agreement with the observation that solitons can exist for a few days.

The case of positive dispersion. We briefly mention here for completness of the properties of Equation (51) the case of $\beta < 0$, which is not realized for either oceanic or atmospheric internal waves, although it may arise in some other physical problems, e.g. for for oblique magneto-acoustic waves in plasmas and thin water layers with surface tension. The latter might be of interest even in the context of this paper for interpretation of some laboratory experiments in rotating devices (e.g., [49, 57]) with thin layers of liquid.

The negative coefficient β changes qualitatively the wave properties. First of all, slow solitary waves may now exist. The reason for this has been already discussed. The phase velocity in this case has a minimum, so that "slow" solitary waves with a speed lying below this minimum can exist. These waves can have both exponential and oscillatory asymptotics; they, of course, contain negative parts to satisfy the zero-mass condition, (46).

It is also worth noting that, besides a single solitary wave, there can exist "multisolitons" consisting of two (as in Figure 8) and, in principle, of a larger number of "elementary" solitary waves. [55, 13]. This admits a transparent interpretation based on the consideration of solitary waves as classical particles interacting through their far field asymptotics ("tails") [24]. Because such a solitary wave has a field minimum, it is equivalent to a "potential trough" in which another solitary wave can have a stable equilibrium position. The same is valid for a system of several solitary waves.

3.4. STRONGLY NONLINEAR WAVES

The motions considered up to now were treated under the assumption of weak nonlinearity, which may impose some limitations on the applicability of quantitative results. Although, in the majority of cases, one could expect the qualitative behavior to be well described by the weakly nonlinear models, there are some phenomena which lie principally beyond the framework of such weakly nonlinear approaches. For instance, we mention the formation of inner vortices in internal waves of large amplitude first observed by Davis and Acrivos [11] in a laboratory experiment (without rotation), in the ocean by Pinkel [62], and in the atmosphere by Christie et al. [9]. These effects are difficult for theoretical treatment and only a few numerical studies are currently available (see, e.g., [71]), again without rotation being taken into account. At the same time, some situations where rotation is essential and nonlinearity is not weak can be treated analytically.

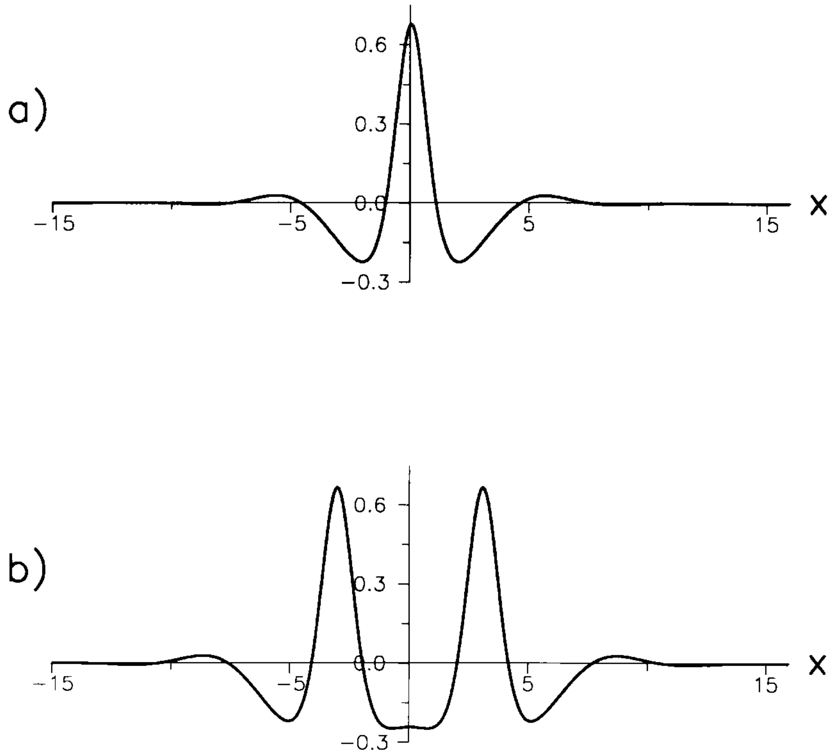

Figure 8. Single- (*a*) and bi-soliton (*b*) solutions of the steady-state rKdV-Equation (57) with positive dispersion ($b = 1,\ \varepsilon = 1,\ V = 0.4$).

Consider for simplicity a model of surface waves on a rotating thin layer of nonstratified fluid. If the scales of the motions are so large that nonhydrostatic terms can be neglected a system of shallow-water equations can be formally obtained from the Boussinesq equations (31). We stress, however, that in the absence of nonhydrostatic terms [proportional to $(kH)^2$], the one-dimensional system,

$$\begin{aligned} \partial_t \eta + \partial_x[(H+\eta)u] &= 0, \\ \partial_t u + u\partial_x u - fv + g\partial_x \eta &= 0, \\ \partial_t v + u\partial_x v + fu &= 0 \end{aligned} \tag{68}$$

is exact for arbitrary nonlinearity. This should be understood in the same sense as for the long, shallow-water surface waves: all nonlinear terms can be retained when we are describing sufficiently long waves provided they remain gentle, the latter condition being checked *a posteriori*.

The system (68) conserves the potential vorticity $\Pi = (f + \partial_x v)/(H + \eta)$ at the trajectories of fluid particles. A particular class of solutions can be described by assuming that the initial potential vorticity is constant everywhere and equals, say, f/H for a localized wave. In this case, the substitution

$$\eta = \frac{H}{f}\partial_x v, \qquad u = -\frac{\partial_t v}{f + \partial_x v} \tag{69}$$

exactly reduces the three nonlinear equations (68) to a single equation in terms of the transverse velocity, [67, 68, 50]

$$\partial_t^2 v - c_0^2 \partial_x^2 v + f^2 v = \partial_t \left[\frac{\partial_t v \partial_x v}{f + \partial_x v}\right] + \frac{f}{2}\partial_x \left[\frac{\partial_t v}{f + \partial_x v}\right]^2. \tag{70}$$

When f tends to zero, the relations (69) and equation (70) lose their validity, as the transverse component of velocity then becomes independent of the other variables and can be put equal to zero. It is of interest to note that a similar exact consequence of the nonrotational shallow water equations can be derived in terms of an auxiliary function ϕ through relations identical to (69) where v is replaced by ϕ and f is put equal to unity,

$$\partial_t^2 \phi - c_0^2 \partial_x^2 \phi = \partial_t \left[\frac{\partial_t \phi \partial_x \phi}{1 + \partial_x \phi}\right] + \frac{1}{2}\partial_x \left[\frac{\partial_t \phi}{1 + \partial_x \phi}\right]^2. \tag{71}$$

Without rotation the evolution of nonlinear waves can be studied directly within (68), which becomes a set of classical shallow water equations or within (71) using Riemann invariant technique elaborated in gas dynamics [41, 74] [it is well known that in this case the equations (68) are equivalent to the gas dynamics equations with the adiabatic index equal to 2]. Rotation brings new qualitative effects which have not yet been described analytically: it provides solutions without shocks in the similar manner as for the model rKdV equation. This is examined in the next subsection where it is shown that there exists a family of steady waves within the framework of (70).

3.4.1. *Stationary waves without high-frequency dispersion*

Let us seek a solution of (70) in the form of a steady progressive wave which depends on $\xi = (x - Vt)/H$. Introducing new dimensionless variable

$$\zeta(\xi) = 1 + \frac{\partial_\xi v(\xi)}{Hf} = 1 + \frac{\eta(\xi)}{H},$$

which has an obvious physical meaning, viz. the normalized total depth of the layer, we reduce (70) to the form,

$$d_\xi^2 \left(\zeta + \frac{U^2}{2\zeta^2}\right) = h^2(\zeta - 1), \tag{72}$$

where $U = V/c_0$, $h = H/L_{RO}$ and $L_{RO} = c_0/f$ is the Rossby – Obukhov radius.

Solutions to this equation can be depicted on the phase plane $\zeta, d_\xi\zeta$ using the first integral of (72):

$$\left(1 - \frac{U^2}{\zeta^3}\right)^2 (\partial_\xi \zeta)^2 - h^2 \left(\zeta^2 - 2\zeta + \frac{2U^2}{\zeta} - \frac{U^2}{\zeta^2}\right) = C. \tag{73}$$

The phase plane is qualitatively similar to that shown in Figure 3. There is a one-parameter family of nonlinear periodic waves, depending on the constant C (as well as on the constant parameters U and h), moving faster than c_0, i.e. having $U > 1$. It is impossible to find the wave shape in an explicit analytical form, but they can be expressed in quadratures [68]. For weakly nonlinear waves (72) reduces to (52) with the corresponding solutions, viz. from quasi-sinusoidal waves up to parabolic waves for the limiting configuration. For large-amplitude waves periodic solutions are qualitatively similar to the corresponding solutions of (52), but the limiting configuration wave, which is a matter of a special interest, has a different analytical form,

$$\begin{aligned} \xi - \xi_0 = \frac{\pm 1}{h} \Bigg\{ & \ln 2 \left[\sqrt{\zeta^2 + 2(\mu - 1)\zeta - \mu} + \zeta + \mu - 1\right] \\ & + \frac{\mu}{\zeta}\sqrt{\zeta^2 + 2(\mu - 1)\zeta - \mu} + \mu^{3/2} \arcsin \frac{(\mu - 1)\zeta - \mu}{\zeta\sqrt{\mu^2 - \mu + 1}} \Bigg\}, \end{aligned} \tag{74}$$

where $\mu = U^{2/3}$, and

$$\xi_0 = \frac{1}{h}\left\{\ln 2\left[\sqrt{3\mu(\mu - 1)} + 2\mu - 1\right] - \frac{\pi}{2}\mu^{3/2}\right\}.$$

The dependence of the phase velocity U on the dimensionless wavelength Λ for the limiting configuration wave is given by the nonlinear dispersion relation

$$\begin{aligned} \frac{\Lambda h}{2} = {} & \ln \frac{\sqrt{3\mu(\mu - 1)} + 2\mu - 1}{\sqrt{\mu^2 - \mu + 1}} + \sqrt{3\mu(\mu - 1)} \\ & + \mu^{3/2}\left(\arcsin \frac{\mu - 2}{\sqrt{\mu^2 - \mu + 1}} + \frac{\pi}{2}\right). \end{aligned}$$

In Figure 9 the forms of the limiting waves are presented for two values of the parameter U. When $U \to 1$ all above formulas naturally reduce to the corresponding formulas of Subsection 3.3 (see **3.3.1**), and the limiting wave profile tends to a sequence of parabolic arcs. Note that for the steady-state solutions it follows from (72) that average value over wave period of ζ must be equal to 1. In other words, the mean value of the perturbation $\eta = 0$, although there is no such restriction within the framework of the nonstationary Equation (70).

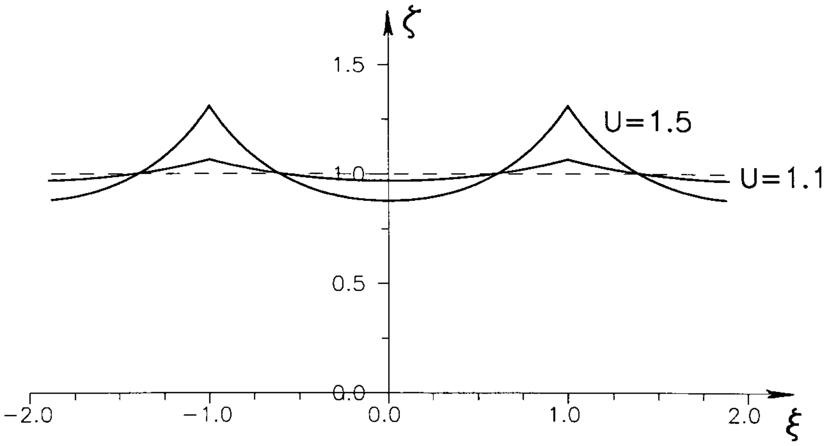

Figure 9. Shapes of the waves of the limiting configuration given by Equation (73) for two different values of the velocity parameter U. The horizontal scale for each wave is normalized by its own half-period λ.

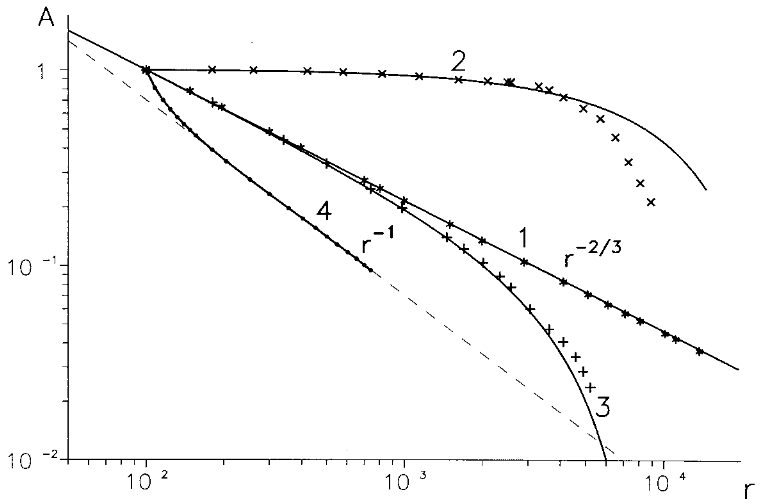

Figure 10. The dependence of the velocity of waves of limiting conviguration shown in Figure 9, on $k = 2\pi/\lambda$ (curve 2). For comparison a corresponding dependence is shown for linear waves (curve 1).

The dependence of wave velocity on wavelength for the limiting configuration waves is shown in Figure 10. In dimensional variables the amplitude of the highest wave (from the crest to the trough) is

$$A \equiv \eta_{max} - \eta_{min} = H(2\mu - 1 - \sqrt{\mu^2 - \mu + 1}),$$

while the maximum and minimum elevations are

$$\eta_{max} = H(\mu - 1), \qquad \eta_{min} = H(\sqrt{\mu^2 - \mu + 1} - \mu),$$

and maximum longitudional and transverse velocities are

$$u_{max} = V\left[1 - \left(\frac{c_0}{V}\right)^{2/3}\right]; \quad v_{max} = c_0\left[\left(\frac{V}{c_o}\right)^{2/3} - 1\right]^{3/2}.$$

The ratio

$$\frac{u_{max}}{v_{max}} = \sqrt{\frac{\mu}{\mu - 1}} > 1,$$

and so, for weak rotation ($\mu \to 1$) the main motion in the water is weakly nonlinear and directed along the x-axis, while for strong nonlinear waves ($\mu \gg 1$) rotation prevails and both components of fluid velocity have the same order.

When $\Lambda \to 0$ ($U \to 1$) the asymptotics take the form $A \propto \Lambda^2$, $V - c_0 \propto \Lambda^2$ and in the opposite limit, $\Lambda \to \infty$ ($U \to \infty$), $A \propto \Lambda^{2/3}$, $V \propto \Lambda$. It is interesting to note that the minimum value of such a wave tends to $-H/2$ when $V, \Lambda \to \infty$, i.e. the level of the free surface of these waves does not fall lower than the half-depth of the basin.

An important feature of the limiting wave is the angle at the crest, φ. We shall characterise it by the the supplementary angle $\psi = \pi - \varphi$:

$$\psi = 2\tan^{-1}\left(h\sqrt{\frac{\mu(\mu - 1)}{3}}\right).$$

This angle is always very small for real oceanic conditions. To estimate it let us put $f = 10^{-4}\,s^{-1}$, $H = 4\,km$, $V/c_0 = 2$, than we obtain $\psi \simeq 6 \cdot 10^{-7}\,rad$. Amplitudes of such waves are quite moderate for wavelength of order of hundred kilometers. For example, for a wave of length $\lambda = 100\,km$ amplitude $A \simeq 2.2\,m$, but for $\lambda = 500\,km \quad A \simeq 40\,m$.

As all these formulae are also valid for internal waves in a two-layer fluid so that analogous estimations can be obtained in the straightforward manner.

One of the interesting features of the (73) is that it belongs to a class of equation with "saturated" nonlinearity. Indeed, it follows from this equation for the perturbations with $\zeta \gg U^2/2$,

$$d_\xi^2 \zeta = h^2 \zeta. \tag{75}$$

Solutions to this equation are obviously proportional to $e^{\pm h\xi}$, so that the asymptotics of the limiting configuration for large-amplitude waves near the crests are exponentials instead of parabolas (compare Figure 4 and Figure 10).

3.4.2. *Influence of the non-hydrostatic dispersion*

In so far Equation (70) is "exact" for waves of infinite wavelengths, we can generalize it taking into account small nonhydrostatic dispersion. As it follows from the discussion after relation (9) both dispersions are of the same order for surface waves in the open ocean having spatial scales $\lambda \simeq 50\,km$. Also for waves of the limiting configuration in the vicinity of each crest it is necessary to take into account non-hydrostatic dispersion because here higher order derivatives play a dominant role. So if we consider a first order correction to this equation due to the parameter H/λ we get the following generalization of (70) [68]:

$$\partial_t^2 v - c_0^2 \partial_x^2 v + f^2 v - \frac{H^3}{3}\partial_x^2 \partial_t^2 v = \partial_t \left[\frac{\partial_t v \partial_x v}{f + \partial_x v} \right] + \frac{f}{2}\partial_x \left[\frac{\partial_t v}{f + \partial_x v} \right]^2 . \quad (76)$$

This equation is more complicated for theoretical analysis and so far there have been no analytical results. For steady waves it can be presented as a Hamiltonian system of two degrees of freedom, which is similar to (55), (56) [55]. It has been rigorously proved [13] that within the framework of this equation there are no stationary solitary solutions, either localized or of kink types. Apparently, its periodic solutions are qualitatively similar to the corresponding solutions of (51) at least in the small-amplitude limit, but they have not been constructed yet either analytically or numerically.

3.5. Two-dimensional effects

The theory presented up to now was concerned with one-dimensional models. Indeed, most of the available observations indicate that waves of the scales under consideration typically have transverse scales much larger than the characteristic wavelength. However a certain curvature of the wave fronts is often seen on satellite or aircraft images. This is most likely a consequence of radial spreading, although there is also a class of situations where transverse structure might be due to lateral boundary conditions or bottom relief features. To describe such *weakly two-dimensional* effects the following extension of the one-dimensional theory proves to be sufficient in most cases.

One can infer all the features of two-dimensional dynamics from the Boussinesq-type equations (31) or (38) for waves long in comparison with the total depth (KdV type waves). For the Benjamin – Ono type waves, i.e. for waves long compared to one of the characteristic layers and short compared to the total depth, the derivation of such a system is also straightforward [26].

First we consider a rather typical case when the waves are generated by a localized source. At large distances from the source these become divergent cylindrical waves. Assuming that the cylindrical wave front has small curvature, of the order of the nonlinear and dispersive terms, one can reduce the two-dimensional problem to an effectively one-dimensional one. In the absence of rotation the wave propagation is described by the exactly solvable cylindrical KdV equation [12, 7, 48] which can be generalized to take rotation into account,

$$\partial_t \left[\partial_r \eta + \alpha \eta \partial_t \eta + \beta \partial_t^3 \eta + \frac{\eta}{2r} \right] = \delta \eta \tag{77}$$

However, taking into account rotation breaks down the integrability. Moreover, Equation (77) as well as the corresponding nonrotational equation, lacks the key element used in our previous asymptotic analysis, viz. strongly localized solitary wave solutions. Moreover, there are no periodic steady solutions either. However, in the case when the "cylindrical factor", i.e. the last term in the brackets, is much smaller than the nonlinear and dispersive terms, it creates a slow variation of a quasisteady solution in the same manner as in the absence of rotation where the wave amplitude decays as $r^{-2/3}$ (see, for instance, [70, 73] and references therein).

If we consider both the rotation and geometric spreading as small perturbation factors we can find the wave amplitude decay as a function of distance r in a similar manner as in Subsection 3.3.2 (the KdV soliton is chosen as a basic solution of the unperturbed equation). Such an analysis results in the following equation for the slow variation of solitary wave amplitude [63]:

$$\partial_r A = -\frac{2}{3}\frac{A}{r} - 2\delta \sqrt{\frac{12\beta}{\alpha}} A^{1/2}. \tag{78}$$

The solution to this equation is readily found to be,

$$A(r) = r^2 \left[\left(\frac{A_0^{1/2}}{r_0} + \delta \frac{3}{4} \sqrt{\frac{12\beta}{\alpha}} \right) \left(\frac{r}{r_0} \right)^{-4/3} - \delta \frac{3}{4} \sqrt{\frac{12\beta}{\alpha}} \right]^2. \tag{79}$$

It is easy to see from (79) that when the rotation is negligible the wave amplitude $A(r)$ decays as $A_0(r/r_0)^{-2/3}$ [70, 73], but when cylindrical divergence is negligibly small (the limit of a plane solitary wave propagating in a rotating fluid) $A(r) = A_0[1 + \delta\sqrt{12\beta/\alpha A_0}(r_0 - r)]^2$ [cf. (64)].

For comparison we present in Figure 11 from [63] the various laws of amplitude decay versus distance for a solitary-like wave for different relationships between nonlinearity, Coriolis (rotational) and Boussinesq dispersions, and geometric spreading.

The next step in describing slow variations along the transverse direction can be taken by using the so-called small-angle or parabolic approximation (the Kadomtsev – Petviashvili-type equation). For quasiplanar waves, the combined effect of rotaion and of slow transverse variations results in the rotation-modified KP Equation (rKP) [52, 27]

$$\partial_x \left[\partial_t \eta + \alpha \eta \partial_x \eta + \beta \partial_x^3 \eta \right] = \delta \eta - \frac{c_0}{2} \partial_y^2 \eta. \tag{80}$$

One of the issues that equations of the rKP-type can address is whether plane waves remains stable with respect to transverse perturbations, and if not, what

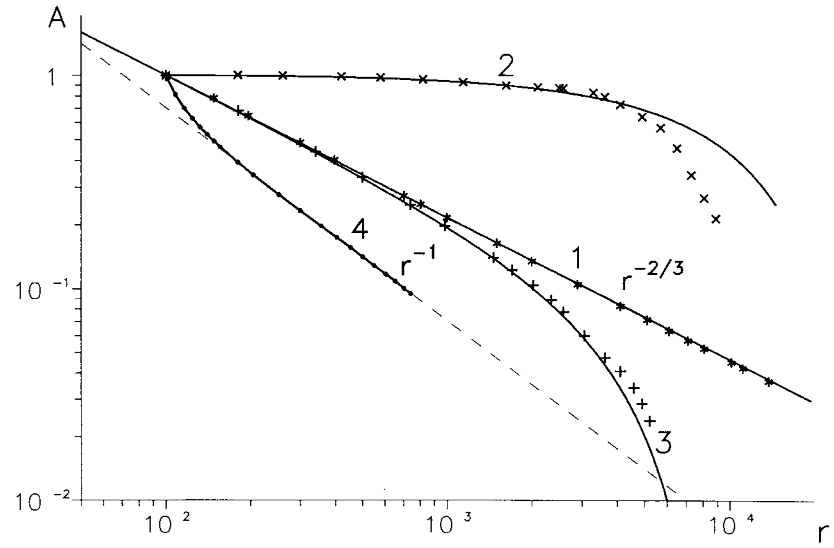

Figure 11. The dependence of the wave amplitude versus distance within the framework of Equation (77). The solid line 1 corresponds to the theory for decay of cylindrical-KdV solitary waves in a non-rotating fluid ($\alpha = 1,\ \beta = 1,\ \gamma = 0$), stars are numerical data; line 2 is the variation of the amplitude of a plane-KdV solitary wave due to the influence of weak rotation ($\alpha = 1,\ \beta = 1,\ \gamma = 10^{-5}$), solid line is theoretical dependence, while crosses are numerical data; line 3 is the synergetic action of the cylindrical divergence and weak rotation on the wave amplitude ($\alpha = 1,\ \beta = 1,\ \gamma = 10^{-5}$), again the solid line corresponds to theory, and pluses are numerical data; and line 4 is the decay of of a linear pulse having the same shape as a KdV soliton under the influence of both types of dispersion and cylindrical divergence ($\alpha = 0,\ \beta = 1,\ \gamma = 10^{-5}$), the solid line is a curve smoothing numerical data (points), the dashed line is an approximation of numerical results for large distances (i.e. intermediate asymptotics).

kind of localized two-dimensional structures may emerge. Although it has not yet been rigorously proved (this can be done using, say, Whitham's technique [74] or some other similar asymptotic approach), there is an evidence that plane waves are stable with respect to long-wave perturbations when dispersion is negative and the rotation is relatively small, while in the range where rotation dominates, transverse instability may occur, resulting in the collapse of the solutions [68]. Where in the parameter space lies the boundary between these radically different regimes is not yet clear and should be further explored. In the case of positive dispersion, plane waves are unstable and strongly localized two-dimensional solitary waves do occur [55].

Most attention has been paid to the numerical investigation of (80) when lateral boundary conditions are imposed by the presence of rigid walls, at which the following boundary condition is fulfilled,

$$\partial_t u + \frac{f}{c_0} u = 0. \tag{81}$$

In linear theory, boundaries with this boundary condition can support Kelvin waves which propagate in the along-channel direction, and decay exponentially away from the boundaries on the scale of the Rossby radius, c_0/f. However, the inclusion of nonlinearity as in the rKP Equation (80) prevents the formation of a steady nonlinear Kelvin wave, and instead any initially localized motion with the structure of a Kelvin wave will decay due to the radiation of Poincare waves (see, for instance, [38, 31, 44]). The mechanism is essentially the two-dimensional counterpart of the one-dimensional radiative decay described above in the Subsection 3.3 (see **3.3.2**).

3.6. Influence of rotation upon small-scale waves

Considering the wave scales and circumstances where rotation is potentially important, one should bear in mind that nonlinearity may lead to specific nonlinear effects for which rotation is important for quite different and, in particular *much smaller*, scales than those estimated so far within the linear theory.

Consider propagation of a modulated internal wave or an isolated wave packet. It is well known that due to cubic nonlinearity the effects of "self-action", such as modulational instability, formation and specific dynamics of wave envelopes inevitably take place. Usually such effects for weakly nonlinear waves can be well described within the framework of the nonlinear Schrödinger-type equations [2, 46, 10]. The mechanism of the self-action is associated, essentially, with the dependence of the period-averaged wave velocity on its amplitude, and its interplay with dispersion effects. The point we would like to emphasize here is that even if rotation effects are negligible at scales of order of the *basic* wavelength, they are likely be important at the scales of the wave packet as a whole, i.e. on the envelope scale and can considerably change the envelope evolution owing to self-modulation. More specifically, the rotation does affect the envelope dynamics if the modulational time scale $\Delta\omega^{-1}$ is comparable with the inertial period, i.e. f^{-1}. In terms of spatial scales this condition reads as

$$L_{modul} = c_{gr}/\Delta\omega = O(c_{gr}/f),$$

where c_{gr} is the *group* velocty of the basic wave.

As the group velocities of high frequency internal waves are often much smaller than the velocity of long waves c_0 of the same mode, the spatial scales of wave packets strongly influenced by rotation can be much smaller than the internal Rossby radius (c_0/f) for the corresponding mode. However, for long waves, described by the Boussinesq equations (31),(38) these values are of the same order. It should be underlined that the effect of rotation for the wave packets of short waves is always due to nonlinearity; it affects a low-frequency component which, in its turn, disturbs the propagation velocity of the carrier wave.

The detailed consideration of the nonlinear dynamics of the small-scale internal waves subject to rotation goes beyond the scope of the present review. We con-

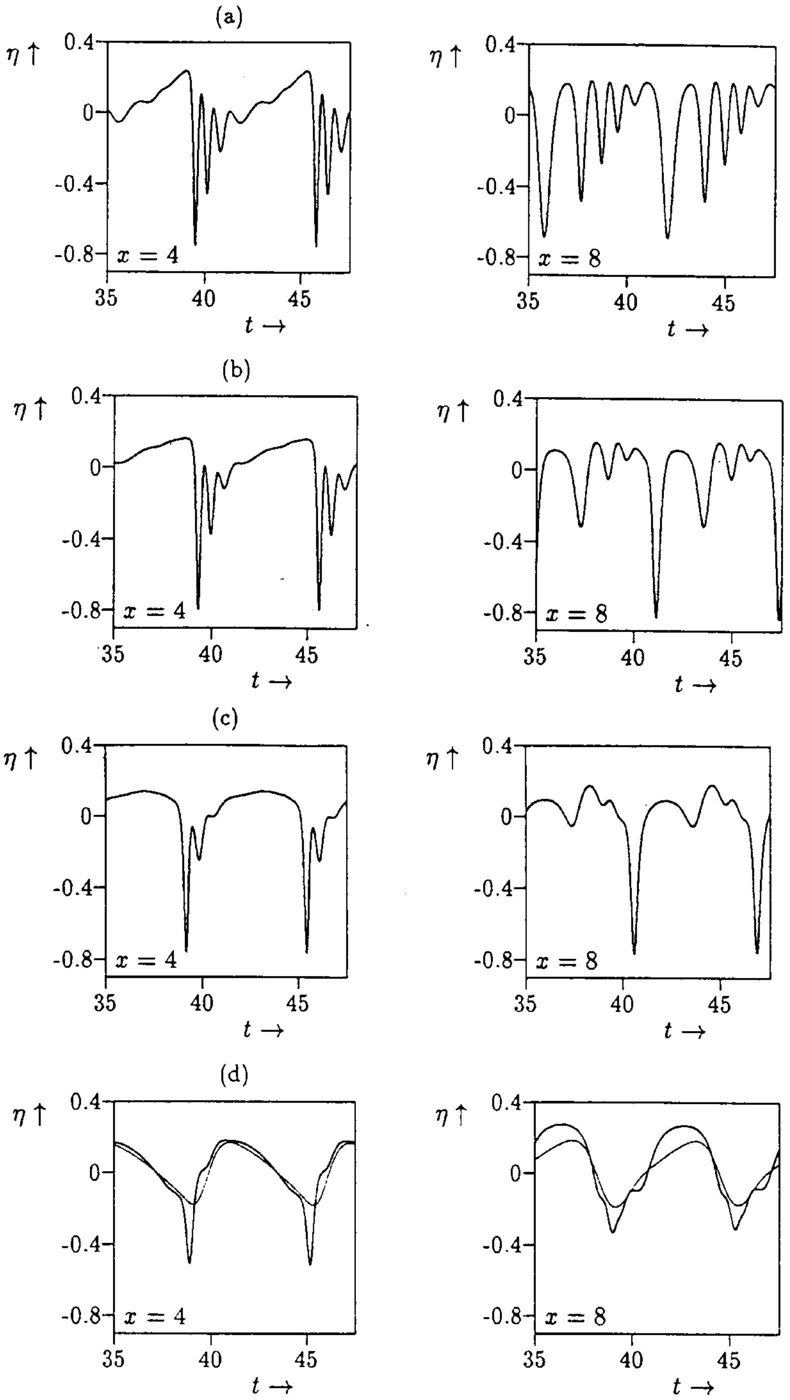

Figure 12. Illustration of the effect of rotation on a nonlinear internal tide. The left column corresponds to a dimensionless distance $x = 4$ from the wave generation region, while the right column to $x = 8$. The Coriolis parameter increases down from (a), where it is equal to zero, through to (d). The thin line in (d) represents a linear solution for which all parameters except for nonlinearity are the same (from [16]).

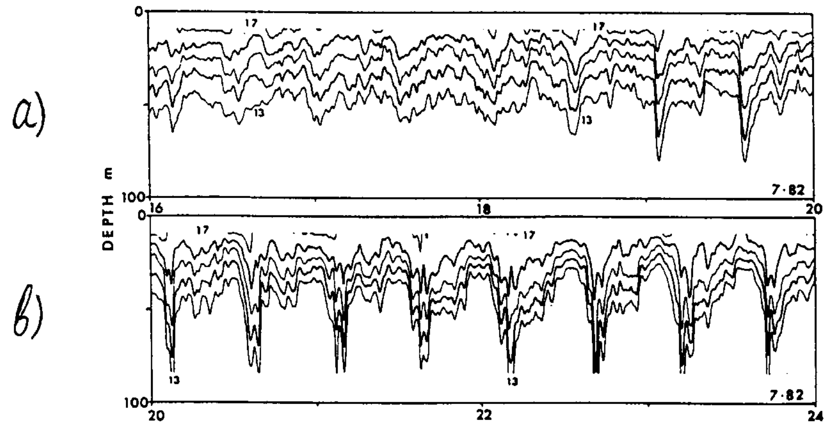

Figure 13. Isotherms plotted from a thermistor chain record of tidal waves in Celtic Sea (July 1983) at about 25 *km* from the shelf break during neap (16 – 18 July) and spring (22 – 24 July) tides (from [61]).

fine ouselves just to a brief outline of specific mechanisms through which rotation affects the envelope nonlinear dynamics.

Modulation of a narrow-band wave train usually occurs through generation of the second harmonics and wave-induced mean flow ("zero" harmonics). In a non-rotating stratified fluid the mean flow is related in a complicated non-local manner with the carrier-wave amplitude [28, 67]. It is easy to see that the presence of rotation inhibits generation of the mean flow, unless the modulation time scale $\Delta\omega$ exceeds greatly the inertial period f. The absence of the mean flows changes qualitatively the character of the wave self-action resulting, in particular, in Schrödinger-type evolution equations.

The above estimates of the wave scales subject to the effect for rotation on the envelope dynamics hold not only for quasi-sinusoidal waves, although an analytic description of such effect on essentially non-sinusoidal waves has not yet been developed. Experimental evidence of such effect and numerical simulations in the context of generation of relatively small-scale internal solitary waves are briefly considered in the next section.

3.7. Internal wave excitation: numerical and experimental data

A large amount of experimental data has been collected about nonlinear internal waves in the ocean and, specifically, about internal solitons (see, e.g., the reviews [54, 4]). Here we shall briefly consider only the data which demostrate an effect of rotation on essentially non-sinusoidal internal waves. For the scales considered, the main source of strong internal waves is most probably associated with the tides; a part of the energy of barotropic tides is transformed into internal waves due to

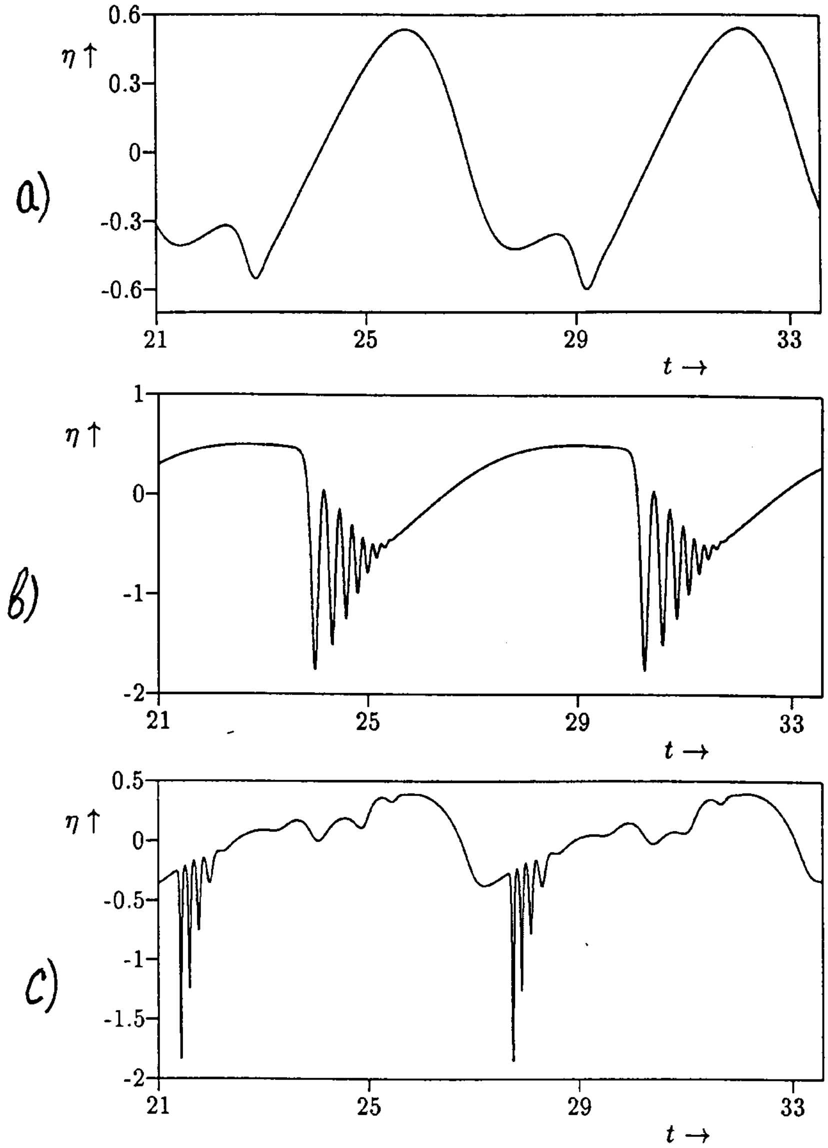

Figure 14. The result of numerical modeling of the interface displacement in a two-layer model corresponding to the experimental situation in the Celtic Sea (Figure 13): *a*) for a neap tide; *b*) for a neap tide without the Earth's rotation; *c*) for a spring tide. One unit in η corresponds to 25 m for the case c) and about 11 m for a) and b) (from [15]).

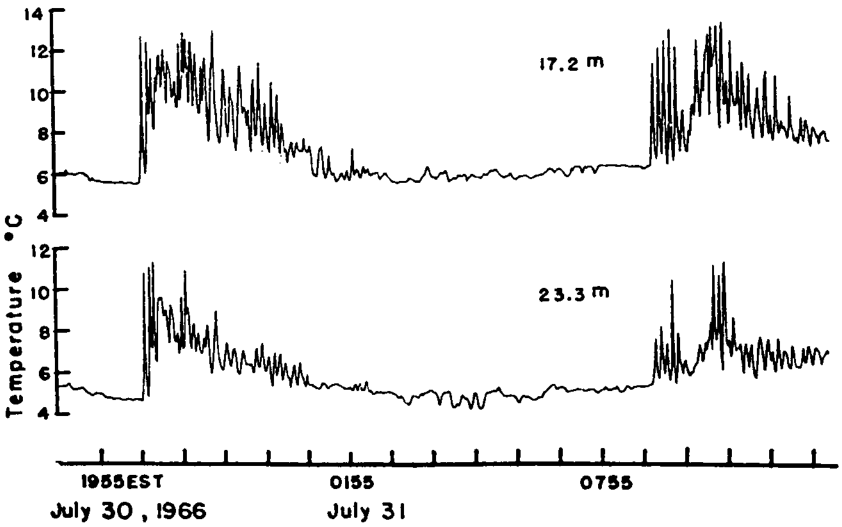

Figure 15. Internal bores generated by a tide in Massachusets Bay. The two temperature records correspond to different depths: 17.2 m and 23.3 m (from [32]).

topographic features (shelf slope, sea mounts, etc.) in the presence of stratification. Typically, the scale of individual solitons are too small to be strongly affected by rotation. However, in the marginal zones of the ocean, they usually exist in groups, as elements of tide-generated internal bores, and the overall scales of these bores can be much more sensitive to rotation. This means that the process of excitation of internal solitons must be considered taking rotation into account. These problems have been addressed by Gerkema [15] (see also [16, 17]), who used a description of the process using a two-layer model of the ocean and considered the rotation-modified Boussinesq equations of the type of (31),(38) (in a slightly different form, for the variables u, v, and η) in the one-dimensional case, with additional terms responsible for a tidal forcing. The wave propagation over a bottom elevation, $H = H(x)$, in the presence of an oscillating barotropic horizontal tidal flow $U = U_0 \sin \omega t$ is considered. The effect of the forcing results, first of all, in the appearance of a term proportional to $U\partial_x H$ in the equation for η which then becomes inhomogeneous (other changes are in terms such as $H\partial_x u$ which become $\partial_x(Hu)$, etc.).

Gerkema performed numerical solutions of this system under the assumption that the elevation is localized and either small compared to the lower layer thickness, or has an arbitrary height but is still smooth enough on the scale of the layer thickness. One of the important observations is that the rotation either decreases the number of solitons in a wave or prevents their formation at all (Figure 12). This result is in agreement with the theoretical considerations given above.

Gerkema then analysed in detail the results of three series of observation of strong tidal-period internal waves made by Pingree and Mardell [61] in the Celtic

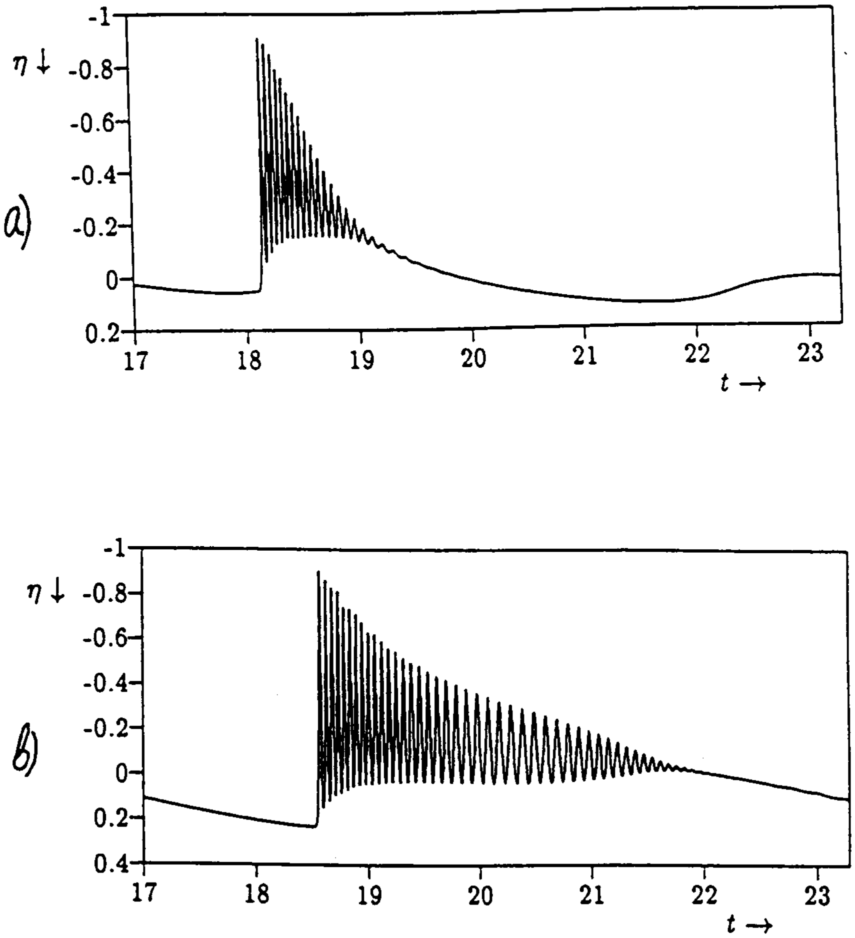

Figure 16. a) The result of numerical modeling of the interface displacement in a two-layer model corresponding to the experimental situation in the Massachusets Bay (Figure 15); b) the same without Earth's rotation (from [15]).

Sea, by Apel et al. [3] in the Sulu Sea, and by Halpern [32] in the Massachuseets Bay (incidentally, the latter is one of the earliest observations of the internal wave solitons). Using the fact that the stratification in such expriments is typically characterized by the presence of a well pronounced thermocline, the author applies the two-layer model mentioned above.

For the experiment [61] (the results of observations are shown in Figure 13) the wave is generated by the internal tide on the shelf slope which is modelled by a linear transition from a deep ($H = 4\,km$) to a shallow ($H = 170\,m$) zone. The results of these computations is shown in Figure 15. It is seen that the soliton amplitudes and their number are in a good agreement with the field observations. It should be emphasized that computation for the non-rotational case (Figure 14a, thin line) gives a much larger number of solitons forming. Thus, the rotation is really important in this case.

Somewhat similar results are obtained for the observations by Halpern [32] (with the use of some later data for the same region, see [33, 8]). Here the number of solitons observed is much larger (Figure 15) but, according to Gerkema, in the non-rotational case the number would be still much greater, so that rotation seems to be important again. Some of these results are shown in Figure 16.

On the contrary, for the experiment of Apel et al. [3], the calculation for the non-rotational KdV-equation give a good agreement with the data. This result could be expected since the near-equatorial position of the Sulu Sea makes the rotational parameter much smaller than in the two other cases (we note that the actual Coriolis parameter $f = 2\Omega \sin\phi$, where ϕ is the local latitude measured from equator).

4. Conclusion

In this review paper we have tried to demonstrate some features of a rather wide class of surface and, especially, internal waves affected by the earth's rotation. From a theoretical viewpoint, these features are associated, to a considerable degree, with a specific evolution Equation (rKdV) which takes into account both effects of rotation and non-hydrostatic dispersion. This equation, being apparently non-integrable, does not admit solitons in an exact sense (in the case of negative dispersion, characteristic of the ocean and atmosphere) but does describe such specific types of waves as a limiting periodic wave of a quasi-parabolic profile, and combinations of KdV-type solitons with such a wave. Also some solutions for strongly nonlinear waves are obtained which is of definite heuristic interest for nonlinear wave theory. Among the recent results, are the description of soliton decay due to rotation, and an investigation of the possibility of a nonlinear-wave breaking in the presence of rotational dispersion.

The observational consequencies of rotation are also non-trivial. While a single internal soliton is usually too narrow to be strongly affected by rotation, the internal bores are much longer so that the process of the *formation* of solitons due to the barotropic tide transformation over the shelf break may depend critically on the rotation, especially for moderate and high latitudes. The computations show much better agreement with the experiment when the rotation is taken into account.

There are still a number of unsolved problems in this area, such as those of the periodic wave evolution with the recurrence of soliton-like structures (observed in numerical experiments), of the stability of steady waves and of multimodal effects. Also we hope that special experiments directed, for instance, at investigating in detail the role of rotation in the formation of internal bores and solitons will be performed. The problems considered might be of interest both for theoreticians involved in nonlinear wave theory, and for oceanographers.

Acknowledgements

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